CMSC 27100 - Summer 2025 — Jonathan Liu
Lecture 9 Worksheet: Combinatorial Proofs

L9

Name:

Problem 1 (Problem Mapping) *Each of the following problems listed is equivalent to another listed problem. Pair them up, and argue that they are equal. You can compute the actual values to help you find the pairs, but your argument should not be that they have the same final solution.*

- Find the number of ways to distribute n identical candies to m distinguishable children.

- Find the number of ways to select m books from a bookshelf with n books in a row, if you can't select two adjacent books.

- Find the number of ways to give n identical candies to m distinguishable kids such that everyone gets at least one and none get more than two. You can assume $2m > n$.

- An ice cream shop has $m - 1$ flavors. Find the number of ways to make a sundae if you want at most n scoops. The order in which you select the flavors doesn't matter.

- Find the number of ways to select positive integers $x_1 \ldots x_m$ to fill in $x_1 + x_2 + \ldots + x_m = n$.

- Find the number of ways to select m people out of n to be on a committee.

- Find the number of ways for n friends to order from a menu of m items if every item must be ordered at least once. The friends are indistinguishable - we only care about the total number of each item ordered.

- Find the number of paths from (0, 0) to (m, n-m) where each step is either up or to the right.

Problem 2 (Combinatorial Proofs) *For each of the following, provide a combinatorial proof. They can all be done by committee-forming, but you may select a different metaphor if you'd prefer.*

(a) Show that

$$\binom{n+1}{k} = \frac{n+1}{k}\binom{n}{k-1}.$$

(b) Show that

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}.$$

(c) Show that

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

Problem 3 (Bonus: Catalan Numbers) *The* ***Catalan numbers*** *are a series of natural numbers that show up all over the place in combinatorics. They follow the recurrence*

$$C_0 = 1, C_n = \sum_{i=1}^{n} C_{i-1}C_{n-i}.$$

They also have a closed form,

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

In this problem, we provide a series of proofs to demonstrate that the recurrence and the closed form are equal.

(a) **Dyck Words** are properly ordered sequences of parentheses. For example, the Dyck words of length 6 are

()()()   (())()   ()(())   ((()))   (()())

Prove that the number of Dyck words with n pairs of parentheses is C_n by strong induction on the claim that it follows the same recurrence.

(b) Consider the number of paths from $(0,0)$ to (n,n) that only travel up or right and never go above the diagonal line $y = x$. For example, the point $(3,3)$ is legal, but the point $(3,4)$ is not. Prove that the number of these paths is equal to the number of Dyck words with n pairs of parentheses.

(c) Prove that the number of paths from $(0,0)$ to (n,n) with the above restriction is equal to $\binom{2n}{n} - \binom{2n}{n+1}$. Hint: reflect the bad paths along the diagonal $y = x + 1$. Where do they end up?

(d) Finally, algebraically show that $\binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n}$, completing the proof.