CMSC 27100 - Summer 2025 — Jonathan Liu
Lecture 9 Worksheet: Combinatorial Proofs

L9.Sol

Problem 1 (Problem Mapping) *Each of the following problems listed is equivalent to another listed problem. Pair them up, and argue that they are equal. You can compute the actual values to help you find the pairs, but your argument should not be that they have the same final solution.*

- Find the number of ways to select m people out of n to be on a committee.
- Find the number of paths from (0, 0) to (m, n-m) where each step is either up or to the right.

  Argument: You must select m steps out of your n total steps to be to the right. This the same as selecting m people out of n total to be on the committee.

- Find the number of ways to distribute n identical candies to m distinguishable children.
- An ice cream shop has $m-1$ flavors. Find the number of ways to make a sundae if you want at most n scoops. The order in which you select the flavors doesn't matter.

  Argument: If you have n identical candies, you can give at most n of them to the first $m-1$ children, and give the rest to child n. Thus, the number of ways to give at most n candies to $m-1$ children is the number of ways to give n candies to m children.

- Find the number of ways to select positive integers $x_1 \ldots x_m$ to fill in $x_1 + x_2 + \ldots + x_m = n$.
- Find the number of ways for n friends to order from a menu of m items if every item must be ordered at least once. The friends are indistinguishable - we only care about the total number of each item ordered.

  Argument: Let x_i be the number of times item i is ordered.

- Find the number of ways to select m books from a bookshelf with n books in a row, if you can't select two adjacent books.
- Find the number of ways to give n identical candies to m distinguishable kids such that everyone gets at least one and none get more than two. You can assume $2m > n$.

  I made a mistake here. For the bottom, note that after giving every kid one piece of candy, you have $n-m$ left over, and you must select m to give them to. Thus the number of ways is $\binom{m}{n-m}$.

  For the top one, consider stars and bars configurations where the bars are the books you take and the stars are the books you don't. Then we have m bars, but we must put a star between each adjacent pair of bars. This means $m-1$ stars must be placed in advance. There are thus m bars (books selected) and $n-m$ stars (books unselected), but $m-1$ are preassigned, so there are $n-2m+1$ stars remaining. By stars and bars, this results in $\binom{n-2m+1+m-1}{m} = \binom{n-m}{m}$.

Problem 2 (Combinatorial Proofs) *For each of the following, provide a combinatorial proof. They can all be done by committee-forming, but you may select a different metaphor if you'd prefer.*

(a) *Show that*

$$\binom{n+1}{k} = \frac{n+1}{k}\binom{n}{k-1}.$$

Solution: *Both are selecting k people from* $n+1$*. The right side is first selecting one person - there are* $n+1$ *ways to choose the first person, but they could have been any of the k people selected, so this overcounts by a factor of k. Thus we have* $n+1$ *ways to pick the first person,* $\binom{n}{k-1}$ *ways to choose the rest, and then we have to correct by k.*

(b) *Show that*

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}.$$

Solution: *We are selecting a committee of r people, and a subcommittee of k of those people. The left side selects the committee first and selects the subcommittee from those people, whereas the right side selects the subcommittee first, and then fills in the rest of the committee.*

(c) *Show that*

$$\binom{2n}{2} = 2\binom{n}{2} + n^2.$$

Solution: *We are selecting two people from* $2n$ *total. Imagine there are n tall people and n short people. Then the right side says we can pick two tall people (*$\binom{n}{2}$*), two short people (*$\binom{n}{2}$*), or one of each (*n^2*).*

Problem 3 (Bonus: Catalan Numbers) *The **Catalan numbers** are a series of natural numbers that show up all over the place in combinatorics. They follow the recurrence*

$$C_0 = 1, C_n = \sum_{i=1}^{n} C_{i-1}C_{n-i}.$$

They also have a closed form,

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

In this problem, we provide a series of proofs to demonstrate that the recurrence and the closed form are equal.

(a) **Dyck Words** are properly ordered sequences of parentheses. For example, the Dyck words of length 6 are

$$()()() \qquad (())() \qquad ()(()) \qquad ((())) \qquad (()())$$

Prove that the number of Dyck words with n pairs of parentheses is C_n by strong induction on the claim that it follows the same recurrence.

Solution: We only present the inductive step here. Note that any Dyck word must start with (, so it can be written as $(a)b$, where a and b are both valid Dyck words as well. Furthermore, a and b have $n-1$ pairs between them. Thus, the number of Dyck words with n pairs of parentheses, by the IH, is

$$\sum_{i=1}^{n} C_{i-1}C_{n-i},$$

where C_{i-1} counts the ways to make a and C_{n-i} counts the ways to make b. We have seen that the recurrence is the same, and the base cases are the same, so there are C_n Dyck words with n pairs.

(b) Consider the number of paths from $(0,0)$ to (n,n) that only travel up or right and never go above the diagonal line $y = x$. For example, the point $(3,3)$ is legal, but the point $(3,4)$ is not. Prove that the number of these paths is equal to the number of Dyck words with n pairs of parentheses.

Solution: Note that at any point, a Dyck word can not have more) than (, but there is no other restriction. This is the same restriction as the one that ensures our path never crosses the line - it can not have more ups than rights.

(c) Prove that the number of paths from $(0,0)$ to (n,n) with the above restriction is equal to $\binom{2n}{n} - \binom{2n}{n+1}$. Hint: reflect the bad paths along the diagonal $y = x+1$. Where do they end up?

Solution: Reflect the bad paths along the diagonal $y = x+1$ after they cross the diagonal. Then they start at $(0,0)$ and end at $(n-1, n+1)$. Note that this never results in an illegal path from $(0,0)$ to $(n-1, n+1)$, and that every path from $(0,0)$ to $(n-1, n+1)$ can be constructed this way because all of these paths must cross the diagonal. Thus, because there are $\binom{2n}{n+1}$ of these paths, there are $\binom{2n}{n+1}$ bad paths.

(d) Finally, algebraically show that $\binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n}$, completing the proof.

Solution:

$$\begin{aligned}
\binom{2n}{n} - \binom{2n}{n+1} &= \binom{2n}{n} - \frac{(2n)!}{(n+1)!(n-1)!} \\
&= \binom{2n}{n} - \frac{(2n)!}{(n+1)\cdot n!\cdot \frac{1}{n}\cdot n!} \\
&= \binom{2n}{n} - \frac{n}{n+1}\cdot\frac{(2n)!}{n!n!} \\
&= \binom{2n}{n} - \frac{n}{n+1}\cdot\binom{2n}{n} \\
&= \frac{1}{n+1}\binom{2n}{n}.
\end{aligned}$$