Lecture 6 Worksheet: Counting Tricks

L6

Name:

Problem 1 (Tricks are for kids!) *Each of the following problems requires at least one of the tricks from class: PIE, Complementary Counting, and Stars and Bars. Some use multiple!*

(a) *How many positive integers less than 100 are a multiple of 3 or 5?*

Solution: *We can use PIE here. Denote by*

$$A = \{x : (3|x) \wedge (0 < x < 100)\},$$

$$B = \{x : (5|x) \wedge (0 < x < 100)\}.$$

Then we know

$$A \cap B = \{x : (3|x) \wedge (5|x) \wedge (0 < x < 100)\} = \{x : (15|x) \wedge (0 < x < 100)\}.$$

We are looking for $|A \cup B|$. By PIE, we know that

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

A has 33 members (3 to 99), B has 19 members (5 to 95), and $A \cap B$ has 6 members (15 to 90), so

$$|A \cup B| = 33 + 19 - 6 = \boxed{46}.$$

(b) *How many positive integers less than 100 are not a multiple of 3 or 5?*

Solution: *The simplest way to solve this problem is to notice that it's the direct complement of the previous problem. There are 99 positive integers less than 100, and 46 of them are multiples of 3 or 5, so $99 - 46 = \boxed{53}$ are not.*

(c) *A painter has been commissioned to paint 5 houses. They have three different colors to choose from, but at least one pair of adjacent houses must have the same color. How many ways are there to paint the houses?*

Solution: *It'll be difficult to deal with the adjacent houses here, so the easiest way around it is to take the complement instead. In that case, we'd like to know the number of ways to paint houses such that no two adjacent ones have the same color. Then there is a decision process through which we can simply paint the houses one at a time, making sure that each house is not the same as the previously-painted one. Then there are 3 choices for the first house and 2 for each of the following ones, leaving $3 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 48$ configurations in total.*

On the other hand, with no restrictions, there are 3 choices for each house so $3^5 = 243$ ways to paint the houses. Thus, in $243 - 48 = \boxed{195}$ ways where at least one pair of adjacent houses have the same color.

(d) *A painter has been commissioned to paint 5 houses. The paint shop has three different colors to choose from, and the painter buys a bucket of paint per house. How many different paint orders could be made? (Two paint orders are the same if they have the same amount of each color of paint, regardless of order)*

Solution: *This problem is equivalent to one where there are 3 bins and 5 indistinguishable balls - each ball represents buying a bucket of that bin's color. By stars and bars, there are $\boxed{\binom{7}{5}}$ configurations.*

(e) *Ten different trees need to be planted in a row. However, if the tallest tree and the shortest tree are next to one another, the shortest tree will not get any sunlight. How many ways are there to plant the trees?*

Solution: *The adjacent condition is difficult to deal with. It's much easier to think about the case where the two trees are adjacent and use* ***complementary counting****. We follow the decision process:*

- *First, select where the tallest and shortest trees go. If they must be adjacent to each other, there are* 9 *pairs of adjacent places for them to be in, and for each pair there are* 2 *options for the order of the trees, for a total of* 18 *options.*
- *Next, place the remaining* 8 *trees in the* 8 *remaining spots - there are* $8!$ *ways to do this.*

Thus, there are $18 \cdot (8!)$ *configurations where the trees are adjacent. There are* $10!$ *configurations in total, leaving* $\boxed{10! - 18 \cdot (8!)}$ *configurations where they are not adjacent.*

Note: *This could have also been counted directly, by counting the number of ways to place two non-adjacent trees, and then placing the rest.*

(f) *Ten identical trees need to be planted in a row. This row spans three blocks. How many ways are there to distribute the trees among the blocks?*

Solution: *This is another classic instance of stars and bars - there are* 10 *indistinguishable balls (trees) to put in one of three bins (blocks). Thus, there are* $\boxed{\binom{12}{10}}$ *distributions of trees.*

(g) *(From AIME 2002) Many states use a sequence of three letters followed by a sequence of three digits as their standard license-plate pattern. Given that each three-letter three-digit arrangement is equally likely, find the probability that such a license plate will contain at least one palindrome (a three-letter arrangement or a three-digit arrangement that reads the same left-to-right as it does right-to-left). Use complementary counting.*

Solution: *We aim to count the number of non-palindrome license plates. For either three-character group, it is a palindrome if the first character matches the last one. Thus, our decision process for each three-character group is as follows:*

- *Select the first character.*
- *Select the second character.*
- *Select the third character to be different from the first.*

This creates $26 \cdot 26 \cdot 25$ *three-letter arrangements and* $10 \cdot 10 \cdot 9$ *three-number arrangements, for a total of* $(26 \cdot 26 \cdot 25) \cdot (10 \cdot 10 \cdot 9)$ *license plates with no palindrome. On the other hand, there are* $26^3 \cdot 10^3$ *license plates in total, so the probability of not having a palindrome is*

$$\frac{26^3 \cdot 10^3 - (26 \cdot 26 \cdot 25) \cdot (10 \cdot 10 \cdot 9)}{26^3 \cdot 10^3} = \frac{26^2 \cdot 10^2 \cdot (26 \cdot 10 - 25 \cdot 9)}{26^3 \cdot 10^3} = \frac{260 - 225}{260} = \boxed{\frac{7}{52}}.$$

(h) *Solve the previous problem using PIE.*

Solution: *We consider the outcomes where there is a letter-based palindrome, a number-based palindrome, and both. A palindrome is formed by the decision process:*

- *Select the first character.*
- *Select the second character.*
- *Select the third character to be the same as the first.*

The first two selections are entirely unconstrained, whereas the last decision only has one choice. Thus there are 26^2 *letter-based palindromes, making* $26^2 \cdot 10^3$ *license plates with a letter-based palindrome. Similarly, there are* 10^2 *number-based palindromes, making* $26^3 \cdot 10^2$ *license plates with a number-based palindrome. However,* $26^2 \cdot 10^2$ *license plates have both palindromes. Therefore, the probability of selecting a license plate with a palindrome is*

$$\frac{26^2 \cdot 10^3 + 26^3 \cdot 10^2 - 26^2 \cdot 10^2}{26^3 \cdot 10^3} = \frac{26^2 \cdot 10^2 \cdot (10 + 26 - 1)}{26^3 \cdot 10^3} = \frac{35}{260} = \boxed{\frac{7}{52}}.$$

Problem 2 (Bonus: Parking Problems) *The first problem is harder than the course is asking for, but is doable with the tricks from this class. The second problem requires quite a bit of creativity - ask for a hint if you're stuck!*

1. *There are 20 parking spots in a row. 14 cars come in, one at a time, and park in a random spot. A monster truck comes in and needs two adjacent spots to park. What is the probability that the monster truck can park?*

   Solution: *We proceed by complementary counting - finding the probability that the monster truck can not park. First, we look for the number of configurations where the monster truck can't park. Converting this to balls and bins, we can actually view this situation as follows: the balls are the empty spots, and each bin represents a space between two cars (or past the ends). For example, bin* 1 *represents the space to the left of the first car, bin* 2 *represents the space between the first car and the second, and so on. There are* 15 *bins in total, and we can only put at most one ball in each bin, or else there will be space for the monster truck to park. Thus, there are* $\binom{15}{6}$ *configurations possible.*

   Now, we must find the number of configurations in total. This, then, is the number of ways to throw 6 *indistinguishable balls into* 15 *bins with no restrictions. We note that by stars and bars this is actually* $\binom{20}{6}$*, which is equivalent to the number of ways to leave* 6 *spots empty, which makes a lot of sense.*

   Thus, by complementary counting, the probability that the monster truck can park is

$$\boxed{\frac{\binom{20}{6} - \binom{15}{6}}{\binom{20}{6}}}.$$

2. *Suppose a one-way street has n parking spaces labeled* $1, 2, \cdots n$*. Each car coming in has a preferred spot, but if it's taken they will keep driving forwards and take the next open spot. If we say that the ith car wants to park in spot* a_i*, then call* $(a_1, a_2, \ldots, a_n)$ *a* ***parking function*** *of length n if all the cars can park. For example, the sequence* $(2, 2, 1)$ *is a parking function because car one parks in spot 2, car two can't park in spot 2 so it parks in spot 3, and car three parks in spot 1. On the other hand,* $(2, 2, 3)$ *is not a parking function because car three will try to park in spot* 3 *but it's already taken by car* 2*, and there are no more spaces after* 3*. Show that for a parking lot of size n, there are* $(n+1)^{n-1}$ *different parking functions.*

   Hint: *The* $n+1$ *gives us a hint that we should think about a situation where there are* $n+1$ *cars instead. A crazier thing to do: make the street a circle instead of a one-way.*

   Hint 2: *In a valid parking function with the above transformation, nobody is parking in spot* $n+1$*.*

   Solution: *Come talk to me about it!*