CMSC 27100 - Summer 2025 — Jonathan Liu

Lecture 3 Pop Quiz: Induction

L3

Name:

Problem 1 (Another Sum) *Prove that the sum of the first n odd numbers is n^2. Namely, prove that for all $n \in \mathbb{N}$,*

$$\sum_{i=1}^{n}(2i-1) = n^2.$$

Problem 2 (Chicken Nuggets) *McDonald's is currently selling chicken nuggets in packages of 3 or 5 nuggets. Let $P(n)$ denote the proposition that you can buy exactly n nuggets. For example, $P(6)$ is true because you can buy two packs of three, and $P(8)$ is true, but $P(7)$ is false. Show that for all $n \geq 8$, $P(n)$ is true. (Hint: your base case will consist of $n = 8, 9, 10$, and your IH will start from $k \geq 11$. Why?)*

You will be writing proofs by induction by following this format. Bold words should not be changed, though parentheses indicate options.

PROBLEM 3 (THE EXAMPLE FROM CLASS) *Prove that the sum of the first n numbers is* $\frac{(n)(n+1)}{2}$. *Namely, prove that for all* $n \in \mathbb{N}$,

$$\sum_{i=0}^{n} i = \frac{(n)(n+1)}{2}.$$

Proof.
We aim to show that for all n, $\sum_{i=0}^{n} i = \frac{(n)(n+1)}{2}$. **We proceed by induction on** n. **Define** $P(n)$ **to mean that** $\sum_{i=0}^{n} i = \frac{(n)(n+1)}{2}$.

Base Case: Our base case(s) here (is/are) $n = 0$. For $n = 0$, note that $\frac{(0)(0+1)}{2} = 0$, so our equation holds.

Inductive Hypothesis: Assume that $\sum_{i=0}^{k} i = \frac{(k)(k+1)}{2}$ **for (**$k = n$ **/ all** $1 \leq k \leq n$**).**

Inductive Step: We aim to show that if the IH is true, then $\sum_{i=0}^{n+1} i = \frac{(n+1)((n+1)+1)}{2}$.
Note that

$$\sum_{i=0}^{n+1} i = \sum_{i=0}^{n} i + (n+1).$$

By the Inductive Hypothesis, this is equal to

$$\begin{aligned}\sum_{i=0}^{n} i + (n+1) &= \frac{(n)(n+1)}{2} + (n+1) \\ &= \frac{(n)(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{(n+2)(n+1)}{2} \\ &= \frac{(n+1)((n+1)+1)}{2}.\end{aligned}$$

Thus, we have shown that $P(n)$ **implies** $P(n+1)$**, so** $\forall n\, P(n)$**.** □

PROBLEM 4 (WATER BALLOON FIGHT) *There are* $2n+1$ *people at a field day, each with a water balloon. When the fight starts, each person throws the balloon at their nearest neighbor. Show that, for any* n, *at least one person is not hit. (Hint: Start by looking at the two people closest to each other.)*

Problem 5 (Eye-landers) *There are 100 people on an island. Each have either orange or purple eyes, but there are no reflective surfaces. The rules of island are strict: if you find out the color of your own eyes, you must leave the island that night. A tourist visiting the island meets all of the villagers, and while leaving, says "wow, I had never seen orange eyes before today!" What happens to the island?*

Problem 6 (En-Tangled) *The witch has captured two princesses and locked them in opposite sides of a tower. They each have a window, through which they can each see perfectly non-overlapping halves of a forest. The witch tells them that there are either 17 or 21 trees in the forest. At 3pm each day, either may guess the number of trees in the forest. If they are right, they may both leave. If not, they are both locked in the tower forever - no second chances. They do not have to guess on any given day though. The princesses may not communicate in any way, though they will know if the other made a guess. How many days does it take for the princesses to go free, and how do they do so?*