L3

Name:

Problem 1 (Another Sum) *Prove that the sum of the first n odd numbers is* $(n+1)^2$*. Namely, prove that for all* $n \in \mathbb{N}$,

$$\sum_{i=1}^{n}(2i-1) = n^2.$$

Proof.
We aim to show that for all n, $\sum_{i=1}^{n}(2i-1) = n^2$. We proceed by induction on n. Define $P(n)$ to mean that $\sum_{i=1}^{n}(2i-1) = n^2$.

Base Case: Our base case(s) here is $n = 1$. For $n = 1$, note that $1 = 1^2$, so our equation holds.

Inductive Hypothesis: Assume $\sum_{i=1}^{n}(2i-1) = n^2$.

Inductive Step: We aim to show that if the IH is true, then $\sum_{i=1}^{n+1}(2i-1) = (n+1)^2$.
Note that

$$\sum_{i=1}^{n+1}(2i-1) = \sum_{i=1}^{n}(2i-1) + (2(n+1)-1).$$

By the Inductive Hypothesis, this is equal to

$$\begin{aligned}\sum_{i=1}^{n}(2i-1) + (2(n+1)-1) &= n^2 + 2n + 2 - 1 \\ &= n^2 + 2n + 1 \\ &= (n+1)^2,\end{aligned}$$

as desired.
Thus, we have shown that $P(n)$ implies $P(n+1)$, so $\forall n\, P(n)$. $\square$

PROBLEM 2 (CHICKEN NUGGETS) *McDonald's is currently selling chicken nuggets in packages of 3 or 5 nuggets. Let $P(n)$ denote the proposition that you can buy exactly n nuggets. For example, $P(6)$ is true because you can buy two packs of three, and $P(8)$ is true, but $P(7)$ is false. Show that for all $n \geq 8$, $P(n)$ is true. (Hint: your base case will consist of $n = 8, 9, 10$, and your IH will start from $k \geq 11$. Why?)*

Proof.
We aim to show that for all $n \geq 8$, we can buy n nuggets. We proceed by **strong** induction on n. Define $P(n)$ to mean that we can buy n nuggets.

Base Case: We'll need three base cases here: $8, 9, 10$. Note that

- 8 = 3 + 5,
- 9 = 3 + 3 + 3,
- 10 = 5 + 5.

Thus, $P(8)$, $P(9)$, and $P(10)$ are all true.
Inductive Hypothesis: Assume that $P(k)$ is true for all $8 \leq k \leq n$.

Inductive Step: We aim to show that if the IH is true, then $P(n+1)$ is true - we can buy exactly $n+1$ nuggets.
Note that $n+1$ nuggets can be bought as a combination of $n-2$ nuggets and 3 nuggets. We can of course buy 3 nuggets, and by the IH we can buy $n-2$ nuggets. Thus, we can buy $n+1$ nuggets.
Thus, we have shown that $P(8) \wedge P(9) \wedge \ldots \wedge P(n)$ implies $P(n+1)$, so $\forall n\, P(n)$. $\square$

You will be writing proofs by induction by following this format. Bold words should not be changed, though parentheses indicate options.

Problem 3 (The Example from Class) *Prove that the sum of the first n numbers is* $\frac{(n)(n+1)}{2}$. *Namely, prove that for all* $n \in \mathbb{N}$,

$$\sum_{i=0}^{n} i = \frac{(n)(n+1)}{2}.$$

Proof.

We aim to show that for all n, $\sum_{i=0}^{n} i = \frac{(n)(n+1)}{2}$. **We proceed by induction on** n. **Define** $P(n)$ **to mean that** $\sum_{i=0}^{n} i = \frac{(n)(n+1)}{2}$.

Base Case: Our base case(s) here (is/are) $n = 0$. For $n = 0$, note that $\frac{(0)(0+1)}{2} = 0$, so our equation holds.

Inductive Hypothesis: Assume that $\sum_{i=0}^{k} i = \frac{(k)(k+1)}{2}$ **for (**$k = n$ **/ all** $1 \leq k \leq n$**).**

Inductive Step: We aim to show that if the IH is true, then $\sum_{i=0}^{n+1} i = \frac{(n+1)((n+1)+1)}{2}$.
Note that

$$\sum_{i=0}^{n+1} i = \sum_{i=0}^{n} i + (n+1).$$

By the Inductive Hypothesis, this is equal to

$$\begin{aligned}\sum_{i=0}^{n} i + (n+1) &= \frac{(n)(n+1)}{2} + (n+1) \\ &= \frac{(n)(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{(n+2)(n+1)}{2} \\ &= \frac{(n+1)((n+1)+1)}{2}.\end{aligned}$$

Thus, we have shown that $P(n)$ **implies** $P(n+1)$**, so** $\forall n\, P(n)$**.** □

Problem 4 (Water Balloon Fight) *There are* $2n+1$ *people at a field day, each with a water balloon. When the fight starts, each person throws the balloon at their nearest neighbor. Show that, for any n, at least one person is not hit. (Hint: Start by looking at the two people closest to each other.)*

Extra Hint: There are two cases: either only those two people throw balloons at each other, or someone else also throws a balloon at one of them two. What happens in each case?

PROBLEM 5 (EYE-LANDERS) *There are 100 people on an island. Each have either orange or purple eyes, but there are no reflective surfaces. The rules of island are strict: if you find out the color of your own eyes, you must leave the island that night. A tourist visiting the island meets all of the villagers, and while leaving, says "wow, I had never seen orange eyes before today!" What happens to the island?*

Hint: In what situation would someone leave on night one? If nobody leaves night one, what does that tell the people on day two?

PROBLEM 6 (EN-TANGLED) *The witch has captured two princesses and locked them in opposite sides of a tower. They each have a window, through which they can each see perfectly non-overlapping halves of a forest. The witch tells them that there are either 17 or 21 trees in the forest. At 3pm each day, either may guess the number of trees in the forest. If they are right, they may both leave. If not, they are both locked in the tower forever - no second chances. They do not have to guess on any given day though. The princesses may not communicate in any way, though they will know if the other made a guess. How many days does it take for the princesses to go free, and how do they do so?*

Hint: Same as the last problem - what do the princesses learn if neither person guesses on day one? What would make them guess on day two?