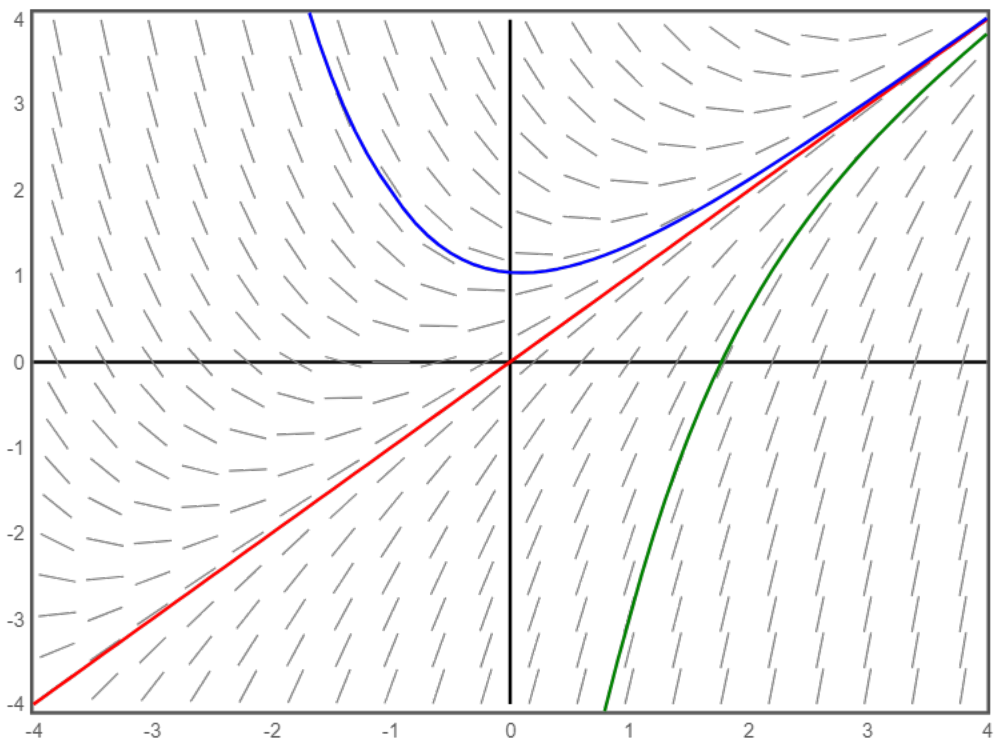
MATH 1B DISCUSSION WORKSHEET - 10/30/18

DIRECTION FIELDS AND EULER'S METHOD

1. Working with Direction Fields



(1) Sketch the solution on this graph corresponding to the initial value $y(1) = -3$.

Drawn in green.

(2) Sketch the solution on this graph corresponding to the initial value $y(-1) = 2$. Use this solution to estimate $y(1)$.

Drawn in blue. $y(1)$ is approximately 1.5.

(3) Sketch the solution on this graph corresponding to the initial value $y(1) = 1$. What line is this?

Drawn in red. This line is the line $y = x$.

2. Euler's Method

(1) Consider the IVP $y' = -xy^2$ with $y(1) = 1$.
  (a) Use Euler's Method to approximate $y(2)$ with step size 0.5.

We'll begin by estimating $y(1.5)$ because our step size is 0.5. In this case, we can see through Euler's Method that

$$y(1.5) = y(1) + (1.5 - 1)y'(1) = 1 + (0.5)(-(1)(1^2)) = 1 - 0.5 = 0.5.$$

Thus, we estimate that $y(1.5) = 0.5$. From here, we can estimate $y(2)$. Through Euler's Method, we can see that

$$y(2) = y(1.5) + (2 - 1.5)y'(1.5).$$

Using the value we found earlier for $y(1.5)$, we plug values in to find

$$y'(1.5) = -(1.5)(0.5^2) = -.375,$$

which means

$$y(2) = 0.5 + (2 - 1.5)(-0.375) = 0.5 - (0.5)(0.375) = 0.3125,$$

so our estimate is $\boxed{y(2) = 0.3125}$.

  (b) Use Euler's Method to approximate $y(2)$ with step size 0.25.

Because we use step size 0.25, we will approximate $y(1.25)$, then $y(1.5)$, then $y(1.75)$, and finally $y(2)$. Writing it out, we can see

$$\begin{aligned}
y(1.25) &= y(1) + (1.25 - 1)y'(1) \\
y(1.25) &= 1 + (0.25)(-(1)(1)^2) \\
y(1.25) &= 0.75 \\
y(1.5) &= y(1.25) + (1.5 - 1.25)y'(1.25) \\
y(1.5) &= 0.75 + (0.25)(-(1.25)(0.75)^2) \\
y(1.5) &= \frac{147}{256} \approx 0.574 \\
y(1.75) &= y(1.5) + (1.75 - 1.5)y'(1.5) \\
y(1.75) &= \frac{147}{256} + (0.25)\left(-(1.5)\left(\frac{147}{256}\right)^2\right) \\
y(1.75) &= \frac{236229}{524288} \approx 0.45 \\
y(2) &= y(1.75) + (2 - 1.75)y'(1.75) \\
y(2) &= \frac{236229}{524288} + (0.25)\left(-(1.75)\left(\frac{236229}{524288}\right)^2\right) \\
y(2) &\approx 0.36175
\end{aligned}$$

(c) Solutions to this differential equation are of the form $y = \dfrac{2}{C + x^2}$ for some constant C. Prove that these are in fact solutions to the differential equation.

Note that the original differential equation was $y' = -xy^2$. We'll find the left side and the right side, and then demonstrate that they are equivalent. First, we'll use the equation for y is find the derivative of y

$$\begin{aligned} y &= \frac{2}{C + x^2} \\ y' &= (-2x)\frac{2}{(C + x^2)^2} \\ y' &= \frac{-4x}{(C + x^2)^2} \end{aligned}$$

On the other hand, we should evaluate xy^2.

$$\begin{aligned} -xy^2 &= -x\left(\frac{2}{C + x^2}\right)^2 \\ &= \frac{-4x}{(C + x^2)^2} \end{aligned}$$

Thus, we can see that $y' = -xy^2$.

(d) Find the value of C that satisfies this IVP.

We know that the curve we choose must contain the point $y(1) = 1$. Plugging that point in, we find

$$\begin{aligned} y &= \frac{2}{C + x^2} \\ 1 &= \frac{2}{C + 1^2} \\ C + 1 &= 2 \\ C &= 1 \end{aligned}$$

Therefore, our final solution to the differential equation is $y = \frac{2}{1+x^2}$.

(e) Plug in $x = 2$ to determine how close your approximations were to the actual value. Which approximation was better?

If we plug $x = 2$ into the actual function, we find

$$y(2) = \frac{2}{1 + (2^2)} = 0.4.$$

Our first approximation with two steps gave $y(2) = 0.3125$, while our second approximation with four steps gave $y(2) = 0.36175$. Thus, we can see that our second approximation, which had more steps, was a better approximation.

(2) Consider the IVP $y' = -\frac{2xy}{1+x^2}$ with $y(0) = 1$.

(a) If I wanted to estimate $y(1)$ in four steps, how big would each step be?

I want to go from $x = 0$ to $x = 1$ in four steps, so the size of each step would be $\frac{1-0}{4} = 0.25$.

(b) Use Euler's Method to approximate $y(1)$ in four steps.

$$\begin{aligned}
y(0.25) &= y(0) + (0.25 - 0)y'(0) \\
y(0.25) &= 1 + (0.25)\left(-\frac{2(0)(1)}{1+(0)^2}\right) \\
y(0.25) &= 1 \\
y(0.5) &= y(0.25) + (0.5 - 0.25)y'(0.25) \\
y(0.5) &= 1 + (0.25)\left(-\frac{2(0.25)(1)}{1+(0.25)^2}\right) \\
y(0.5) &= 1 + \left(-\frac{0.5}{4.25}\right) \\
y(0.5) &= 1 + \frac{-2}{17} \\
y(0.5) &= \frac{15}{17} \\
y(0.75) &= y(0.5) + (0.75 - 0.5)y'(0.5) \\
y(0.75) &= \frac{15}{17} + 0.25\left(-\frac{2(0.5)\left(\frac{15}{17}\right)}{1+\frac{1}{4}}\right) \\
y(0.75) &= \frac{15}{17} - \left(\frac{\left(\frac{15}{17}\right)}{5}\right) \\
y(0.75) &= \frac{15}{17} - \left(\frac{3}{17}\right) \\
y(0.75) &= \frac{12}{17} \\
y(1) &= y(0.75) + (1 - 0.75)y'(0.75) \\
y(1) &= \frac{12}{17} + (0.25)\left(-\frac{2(0.75)\left(\frac{12}{17}\right)}{1+\frac{9}{16}}\right) \\
y(1) &= \frac{12}{17} + \left(-\frac{\frac{3}{2}\left(\frac{12}{17}\right)}{\frac{25}{4}}\right) \\
y(1) &= \frac{12}{17} + \left(-\frac{\left(\frac{18}{17}\right)}{\frac{25}{4}}\right) \\
y(1) &= \frac{12}{17} + \left(-\frac{72}{425}\right) \\
y(1) &= \frac{228}{425} \approx 0.54.
\end{aligned}$$

(c) Solutions to this differential equation are of the form $y = \dfrac{C}{1+x^2}$ for some constant C. Prove that these are in fact solutions to the differential equation.

Once again, we will prove that both sides of the differential equation $y' = -\frac{2xy}{1+x^2}$ end up evaluating to the same expression. First, we note that

$$y' = -2x\frac{C}{(1+x^2)^2}.$$

Meanwhile, we can see that $\frac{-2xy}{1+x^2}$

$$\begin{aligned}\frac{-2xy}{1+x^2} &= -2x\frac{\frac{C}{1+x^2}}{1+x^2}\\ &= -2x\frac{C}{(1+x^2)^2}.\end{aligned}$$

Thus, we can see that the left and right side of the equation both evaluate to the same expression, so they are equal. Thus, these are valid solutions.

(d) Find the value of C that satisfies this IVP.

We know that the point $y(0) = 1$ is on the solution curve. Plugging in, we find $1 = \frac{C}{1+(0)^2}$ so $C = 1$, so our curve is $\frac{1}{1+x^2}$.

(e) Plug $x = 1$ into the solution to determine how close your approximation was to the actual value.

Plugging in, we can see that $y(1) = \frac{1}{1+1} = \frac{1}{2}$, which is quite close to our approximation.