MATH 1B DISCUSSION WORKSHEET - 11/1/18

SEPARABLE DIFFERENTIAL EQUATIONS + ORTHOGONAL TRAJECTORIES AND MIXING PROBLEMS

1. Separable Differential Equations

(1) Determine whether the following differential equations are separable. For each one, if it is separable, rewrite it in the form

$$g(y)dy = f(x)dx.$$

(a) $y' = e^y$

It is. $\frac{dy}{dx} = e^y$ $\boxed{e^{-y} dy = dx}$

(b) $y' = 5y^3 - y^3 \sin(x^2)$

It is. $\frac{dy}{dx} = (5 - \sin(x^2))y^3 \Rightarrow \boxed{y^{-3} dy = (5 - \sin(x^2)) dx}$

(c) $y' + 2xy = 4x$

It is. $\frac{dy}{dy} = 4x - 2xy = 2x(2-y)$ $\boxed{\frac{1}{2-y} dy = 2x\, dx}$

(d) $y' = xy - 3x - 2y + 6$

It is. $\frac{dy}{dx} = (x-2)(y-3) \Rightarrow \boxed{\frac{1}{y-3} dy = (x-2) dx}$

(e) $yy' = e^{2x+4y}$

It is. $\frac{dy}{dx} = e^{2x} \frac{e^{4y}}{y}$ $\boxed{ye^{-4y} dy = e^{2x} dx}$

(2) Solve the following IVP:

$$x\frac{dy}{dx} = 2(y-4), y(1) = 7.$$

$\frac{1}{y-4} dy = \frac{2}{x} dx$

$\ln(y-4) = 2\ln(x) + C$

$y - 4 = e^{2\ln(x) + C}$

$\boxed{y = 4 + x^2 \cdot e^C}$

$y(1) = 7$

$7 = 4 + (1)^2 e^C$

$e^C = 3 \longrightarrow \boxed{y = 4 + 3x^2}$

(3) Solve the following IVP:

$$\frac{dy}{dx} = \frac{x^2 - 1}{xy}, y(1) = 3.$$

$y\, dy = \frac{x^2-1}{x} dx$

$y\, dy = x - \frac{1}{x} dx$

$\frac{1}{2}y^2 = \frac{1}{2}x^2 - \ln(x) + C$

$y^2 = x^2 - 2\ln(x) + C$

$\boxed{y = \sqrt{x^2 - 2\ln|x| + C}}$

$y(1) = 3$

$3 = \sqrt{1 - 2\ln(1) + C}$

$3 = \sqrt{1 + C} \Rightarrow C = 8$

$\boxed{y = \sqrt{x^2 - 2\ln(x) + 8}}$

(4) Consider the following differential equation:

$$\frac{dy}{dx} = 2x(y-5).$$

(a) Find a general form for the solutions to this differential equation.

$\int \frac{1}{y-5}\,dy = \int 2x\,dx$

$\ln(y-5) = x^2 + C$

$y-5 = e^{x^2+C}$

$\boxed{y = 5 + (e^C)(e^{x^2})}$

(b) Note that during our process, we had to integrate $\frac{1}{y-5}$. Normally this would be fine, but it becomes problematic if $y=5$. Do our solution curves ever run into this problem? Why or why not?

No, because the y value only becomes 5 when $e^C e^{x^2} = 0$, but this is impossible for any finite C or x.

(c) Now, let's consider the worst case scenario. Let's say I wanted to find a curve satisfying the above differential equation with the constraint $y(0) = 5$, Suddenly our separation doesn't work anymore! However, that doesn't mean we can't find a curve satisfying the differential equation.

(i) Use Euler's Method and the point $y(0) = 5$ to approximate $y(2)$ with step size 1.

$y(1) = y(0) + (1-0)\,y'(0)$

$y(1) = 5 + 1\,(2(0)(5-5))$

$y(1) = 5$

$y(2) = y(1) + (2-1)\,y'(1)$

$y(2) = 5 + 1(2)(5-5)$

$\boxed{y(2) = 5}$

(ii) What do you notice about the approximation?

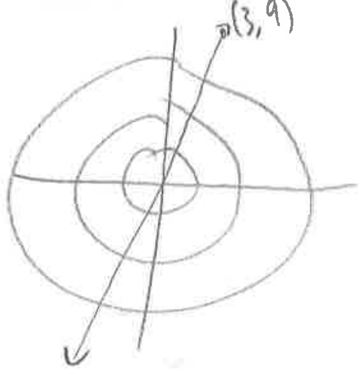
The y-value is always 5.

(iii) Using the approximation, guess what the solution to this IVP might be, and prove that it in fact satisfies the differential equation for all values of x.

The solution is the line $y=5$. On this line, $\frac{dy}{dx} = 2x(5-5) = 0$, so the slope of any point on this line will be 0. The slope of the line $y=5$ is also 0, so the line $y=5$ always satisfies the differential equation.

2. Orthogonal Trajectories

(5) Guess the orthogonal trajectory for the family of curves $x^2 + y^2 = r^2$ containing the point $(3, 9)$ by drawing a picture. Then, determine the curve using the techniques we've learned.



Guess: line passing through origin and $(3,9)$. $= \boxed{y=3x}$

$$\frac{d}{dy}\left[x^2+y^2\right] = \frac{d}{dx}\left[r^2\right]$$

$$2x + 2yy' = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-x}{y} \Rightarrow \text{new } \frac{dy}{dx} = \frac{y}{x}$$

$$\frac{1}{y}\,dy = \frac{1}{x}\,dx \qquad \ln(y) = \ln(x) + C$$

$$y = e^{\ln(x)+C} \Rightarrow \boxed{y = e^C(x)}$$

$$9 = e^C(3) \Rightarrow e^C = 3 \Rightarrow \boxed{y=3x}$$

(6) Determine the orthogonal trajectory for the family of curves $xy + ry = 1$ for constant r.

$$y = \frac{1}{x+r}$$

$$ry = 1 - xy$$

$$r = \frac{1}{y} - \cancel{x}$$

$$y' = \frac{-1}{(x+r)^2}$$

$$y' = \frac{-1}{\left(x+\frac{1}{y}-x\right)^2} = \frac{-1}{\frac{1}{y^2}} = -y^2$$

$$\text{new } y' = \frac{1}{y^2}$$

$$\frac{dy}{dx} = \frac{1}{y^2}$$

$$y^2\,dy = dx$$

$$\frac{1}{3}y^3 = x + C$$

$$y^3 = 3x + C$$

$$\boxed{y = \sqrt[3]{3x+C}}$$

3. Mixing Problems

(7) A tank contains 1000 gallons of water 30 pounds of dissolved salt (which is a lot, I think). Salt water that has a concentration of 0.1 lbs/gal of salt enters the tank at a rate of 50 gal/hour. The solution is kept thoroughly mixed and drains from the tank into a second tank at the same rate. What is the concentration of the salt after 2 hours?

IN: $\left(\frac{50 \text{ gal}}{\text{hour}}\right)\left(\frac{0.1 \text{ lbs}}{\text{gal}}\right) = \frac{5 \text{ lbs}}{\text{hour}}$

OUT: $\left(\frac{50 \text{ gal}}{\text{hour}}\right)\left(\frac{S \text{ lbs}}{1000 \text{ gal}}\right) = \frac{\frac{S}{20} \text{ lbs}}{\text{hour}}$

$$\frac{dS}{dt} = 5 - \frac{S}{20}$$

$$\frac{1}{5-\frac{S}{20}}\, dS = dt$$

$$\frac{20}{100-S}\, dS = dt$$

$$-20\ln(100-S) = t + C$$

$$\ln(100-S) = \frac{-t}{20} + C$$

$$100 - S = e^{\frac{-t}{20}+C}$$

$$S = 100 - e^{\frac{-t}{20}+C}$$

$S(0) = 30$

$30 = 100 - e^{C}$

$e^{C} = 70$

$S = 100 - e^{C}\cdot e^{-t/20}$

$$\boxed{S = 100 - 70e^{-t/20}}$$

$S(2) = 100 - 70e^{-1/10}$

Concentration at 2hr. is

$$\boxed{\frac{100 - 70e^{-1/10}}{1000}}$$

(8) That second tank originally started out empty, and drains its mixture at a rate of 30 gallons per hour. Write a differential equation that can be used to model the salt content in the second tank.

IN: $\left(\frac{50 \text{ gal}}{\text{hour}}\right)\left(\frac{100 - 70e^{-t/20}}{1000}\right) = \frac{100 - 70e^{-t/20}}{20} = 5 - 3.5e^{-t/20}$

OUT: Volume$(t) = 0 + 50t - 30t = 20t$

: $\left(\frac{30 \text{ gal}}{\text{hour}}\right)\left(\frac{S}{20t}\right) = \frac{3S}{2t}$

$$\boxed{\frac{dS}{dt} = 5 - 3.5e^{-t/20} - \frac{3S}{2t}}$$