MATH 1B DISCUSSION WORKSHEET - 11/6/18

POPULATIONS AND LINEAR DIFFERENTIAL EQUATIONS

1. Linear Differential Equations

(1) Determine whether the following differential equations are separable, linear, or both. Then, solve each one.

(a) $xy' + 2y = x^2 - x$

This equation is **linear**. Rewriting it into standard form, we find

$$\begin{aligned} xy' + 2y &= x^2 - x \\ y' + \frac{2}{x}y &= x - 1, \end{aligned}$$

so we can see that $p(x) = \frac{2}{x}$. The integrating factor is therefore $e^{\int (2/x)dx} = e^{2\ln(x)} = x^2$, so if we multiply the entire equation by x^2, we find

$$\begin{aligned} y' + \frac{2}{x}y &= x - 1 \\ x^2y' + 2xy &= x^3 - x^2 \\ (x^2y)' &= x^3 - x^2 \\ x^2y &= \frac{1}{4}x^4 - \frac{1}{3}x^3 + C \\ y &= \boxed{\frac{1}{4}x^2 - \frac{1}{3}x + Cx^{-2}} \end{aligned}$$

(b) $y' + 3xy = 6x$

This equation is **both separable and linear**. It is already in linear form, but it can be rewritten as

$$y' = -3xy + 6x = 3x(2 - y),$$

from which we will separate.

$$\begin{aligned} \frac{dy}{dx} &= 3x(2 - y) \\ \frac{1}{2 - y}dy &= 3xdx \\ -\ln|2 - y| &= \frac{3}{2}x^2 + C \\ \ln|2 - y| &= -\frac{3}{2}x^2 + C \\ |2 - y| &= e^{-\frac{3}{2}x^2 + C} \\ 2 - y &= \pm e^{-\frac{3}{2}x^2 + C} \\ y &= \boxed{\pm e^{-\frac{3}{2}x^2 + C} + 2} \end{aligned}$$

We could have solved this as a linear equation as well, but those are generally more work.

(c) $y' + 3xy = 6$ This is also already in the form of a **linear** equation. However, unlike

the previous differential equation, this one is non-separable. It is, however, already in standard form, so we can find the integrating factor:

$$I(x) = e^{\int 3x dx} = e^{\frac{3}{2}x^2}.$$

Multiplying the original differential equation by the integrating factor, we find

$$\begin{aligned} y' + 3xy &= 6 \\ e^{\frac{3}{2}x^2}y' + e^{\frac{3}{2}x^2}3xy &= 6e^{\frac{3}{2}x^2} \\ (e^{\frac{3}{2}x^2}y)' &= 6e^{\frac{3}{2}x^2} \\ e^{\frac{3}{2}x^2}y &= \int 6e^{\frac{3}{2}x^2}dx. \end{aligned}$$

We actually don't have the tools in this class to integrate the expression on the right. Oops.

(d) $xy' - 2y = -8$ Rewriting, we can see that this equation can be written as

$$y' - \frac{2}{x}y = -\frac{8}{x}$$

or as

$$\frac{dy}{dx} = \frac{2y-8}{x},$$

making it **both separable and linear**. We will solve it as a separable differential equation. Rearranging, we find

$$\begin{aligned} \frac{dy}{dx} &= \frac{2y-8}{x} \\ \frac{1}{2y-8}dy &= xdx \\ \int \frac{1}{2y-8}dy &= \int xdx \\ \frac{1}{2}\ln|y-8| &= \frac{1}{2}x^2 + C \\ \ln|y-8| &= x^2 + C \\ |y-8| &= e^{x^2+C} \\ y &= \boxed{8 \pm e^{x^2+C}} \end{aligned}$$

(e) $y' - \frac{1}{2}y = \sin(2t)$

This equation is clearly **linear**, and can not be separated. We find that the integrating factor is

$$I(t) = e^{\int -(1/2)dt} = e^{-\frac{1}{2}t}.$$

Multiplying the differential equation by this integrating factor, we can solve:

$$\begin{aligned}
y' - \frac{1}{2}y &= \sin(2t) \\
e^{-\frac{1}{2}t}y' - e^{-\frac{1}{2}t}\frac{1}{2}y &= e^{-\frac{1}{2}t}\sin(2t) \\
(e^{-\frac{1}{2}t}y)' &= e^{-\frac{1}{2}t}\sin(2t) \\
e^{-\frac{1}{2}t}y &= \int e^{-\frac{1}{2}t}\sin(2t)dt
\end{aligned}$$

[Integration by parts: $u = \sin(2t), du = 2\cos(2t)dt, v = -2e^{-\frac{1}{2}t}, dv = e^{-\frac{1}{2}t}]dt$

$$\begin{aligned}
e^{-\frac{1}{2}t}y &= \sin(2t)\left(-2e^{-\frac{1}{2}t}\right) - \int -2e^{-\frac{1}{2}t}2\cos(2t)dt \\
e^{-\frac{1}{2}t}y &= -2\sin(2t)e^{-\frac{1}{2}t} + 4\int e^{-\frac{1}{2}t}\cos(2t)dt
\end{aligned}$$

[Integration by parts: $u = \cos(2t), du = -2\sin(2t)dt, v = -2e^{-\frac{1}{2}t}, dv = e^{-\frac{1}{2}t}dt$]

$$\begin{aligned}
e^{-\frac{1}{2}t}y &= -2\sin(2t)e^{-\frac{1}{2}t} + 4\left[\cos(2t)\left(-2e^{-\frac{1}{2}t}\right) - \int -2e^{-\frac{1}{2}t}(-2\sin(2t))dt\right] \\
e^{-\frac{1}{2}t}y &= -2\sin(2t)e^{-\frac{1}{2}t} + 4\cos(2t)\left(-2e^{-\frac{1}{2}t}\right) - 16\int e^{-\frac{1}{2}t}\sin(2t)dt \\
e^{-\frac{1}{2}t}y &= -2\sin(2t)e^{-\frac{1}{2}t} - 8\cos(2t)\left(e^{-\frac{1}{2}t}\right) - 16\left(e^{-\frac{1}{2}t}y + C\right) \text{ [Why?]} \\
17e^{-\frac{1}{2}t}y &= -2\sin(2t)e^{-\frac{1}{2}t} - 8\cos(2t)\left(e^{-\frac{1}{2}t}\right) + C \\
y &= \frac{1}{17}\left[-2\sin(2t) - 8\cos(2t)\right] + Ce^{\frac{1}{2}t}
\end{aligned}$$

(f) $xy' + 2y = 12x^2$

This equation can be rewritten as

$$y' + \frac{2}{x}y = 12x,$$

making it **linear**. It is not separable. We find that the integrating factor is

$$I(x) = e^{\int (2/x)dx} = e^{2\ln(x)} = x^2,$$

so if we multiply the entire equation by x^2, we find

$$\begin{aligned} y' + \frac{2}{x}y &= 12x \\ x^2y' + 2xy &= 12x^3 \\ (x^2y)' &= 12x^3 \\ x^2y &= 3x^4 + C \\ y &= \boxed{3x^2 + Cx^{-2}} \end{aligned}$$

(2) Using the last differential equation, find a solution for

$$xy'' + 2y' = 12x^2.$$

Note that this differential equation is the exact same as the last one, except with y'' and y' in the place of y' and y, respectively. If we make the substitution $z = y'$, we can see that this equation becomes

$$xz' + 2z = 12x^2,$$

which has solution $z = 3x^2 + Cx^{-2}$. Thus, the solution to this equation is

$$y' = 3x^2 + Cx^{-2},$$

so

$$\boxed{y = x^3 + -Cx^{-1}}.$$

(3) Show that for some **homogeneous** differential equation

$$\frac{dy}{dx} + p(x)y = 0,$$

if $y = f(x)$ and $y = g(x)$ are both valid solutions, then $y = f(x)+g(x)$ is a valid solution as well.

Plugging in directly, we can see that

$$\begin{aligned} \frac{dy}{dx} + p(x)y &= \frac{d}{dx}[f(x) + g(x)] + p(x)(f(x) + g(x)) \\ &= f'(x) + g'(x) + p(x)f(x) + p(x)g(x) \\ &= f'(x) + p(x)f(x) + g'(x) + p(x)g(x) \\ &= \left[\frac{df(x)}{dx} + p(x)f(x)\right] + \left[\frac{dg(x)}{dx} + p(x)g(x)\right] \\ &= 0 + 0 \\ &= 0, \end{aligned}$$

as desired.

2. Population Models

(1) Suppose we have a pond that will support 1000 fish, and the initial population is 100 fish. Write out a logistic differential equation for this scenario, and determine its solution.

We know that logistic equations are of the form

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right),$$

where M is the carrying capacity. Thus, we can see that

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{1000}\right), P(0) = 100$$

is our IVP. Next, we know that the solution to this differential equation (which we can find through separation or by looking in the textbook) is

$$P(t) = \frac{M}{1 + Ae^{-kt}}, A = \frac{M - P_0}{P_0}.$$

Plugging in M and P_0, we find

$$A = \frac{1000 - 100}{100} = 9,$$

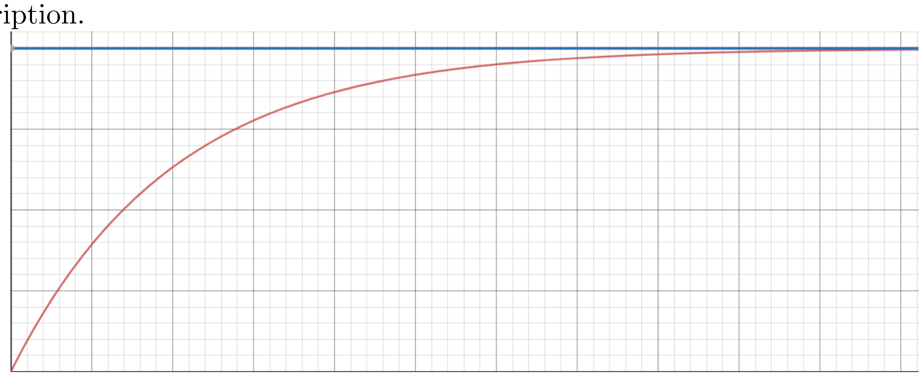
so our final solution is

$$\boxed{P(t) = \frac{1000}{1 + 9e^{-kt}}}.$$

(2) Psychologists in learning theory study learning curves, the graphs of the performance function $P = P(t)$ of someone learning a skill as a function of the training time t. If M represents the maximal level of performance, it is noted that for a certain skill the learning is at first rapid, and then it tapers off (the rate of learning decreases) as $P(t)$ approaches M.

(a) Explain why solutions to the differential equation $\frac{dP}{dt} = k(M - P)$, k a positive constant, fulfill the above description of P. Sketch a typical learning curve.

While P is low, $M - P$ is high, which means that $\frac{dP}{dt}$ is high, as expected. On the other hand, as P approaches M, $P - M$ decreases significantly, so the rate of learning decreases as well. Both of these are expected behaviors that match the description.



(b) What's the relationship between this differential equation and the logistic population growth model? This growth model is similar to logistic population growth,

except that growth is determined solely by distance from M instead of both by distance from M as well as P. As such, the starting growth rate for this model is much higher than that of the logistic growth model.

(c) (c) Suppose that, for a specific learning activity, it is determined that $P_0 = .1M$ and $k = 0.05$ for t measured in hours. How long does it take to reach 90% of M, the maximal level of performance? Using the information we are given, we can start by

rewriting the problem as an IVP:

$$\frac{dP}{dt} = 0.05(M - P), P(0) = 0.1M.$$

This differential equation is separable, so we can solve:

$$\begin{aligned}\frac{1}{M-P}dP &= 0.05dt \\ -\ln|M-P| &= 0.05t + C \\ \ln|M-P| &= -0.05t + C \\ |M-P| &= e^{-0.05t+C} \\ M-P &= \pm e^{-0.05t+C}\end{aligned}$$

At this point, while the solution to the differential equation is technically either positive or negative, we know from context that $M - P$ is always positive. If $M - P$ was negative, then $P > M$, which means that the person has learned more than the maximal level of performance, which is impossible. Thus, we can continue with only the positive solution:

$$\begin{aligned}M - P &= e^{-0.05t+C} \\ P &= M - e^{-0.05t+C}.\end{aligned}$$

We can plug the initial value in to find

$$\begin{aligned}0.1M &= M - e^{C} \\ 0.9M &= e^{C},\end{aligned}$$

which leads us back to

$$P = M - 0.9Me^{-0.05t}.$$

We'd like to find when $P = 0.9M$, so plugging this in, we find

$$\begin{aligned}0.9M &= M - 0.9Me^{-0.05t} \\ 0.1M &= 0.9Me^{-0.05t} \\ \frac{1}{9} &= e^{-0.05t} \\ t &\approx 43.94,\end{aligned}$$

so it takes approximately $\boxed{\text{43.94 hours}}$.