MATH 1B MOCK FINAL

BUT SHORTER AND WITHOUT CHAPTER 17

(1) A 400L swimming pool starts currently holds 200 liters of water as well as 500 milligrams of chlorine. Clean water flows in at a rate of 5 liters per minute, and the mixed water flows out at 3 liters per minute.

Swimming pools are considered safe when the chlorine concentration is below 1mg/L. Will the swimming pool reach a safe level of chlorine before it overflows? Justify your answer.

We will begin by writing a differential equation to model this situation. Note that the amount of the substance going into the solution is actually 0, as only clean water is going in. Thus, the differential equation should be written purely as

$$\frac{dS}{dt} = -(\text{rate of substance leaving the pool}).$$

The rate at which the substance leaves the pool is equal to 3L/min times the concentration of the substance, which is the difficult part to find due to the fact that the volume of the pool changes with time. Note that concentration is equal to amount/volume. To find the volume over time, note that there are 5 liters flowing in per minute but only 3 liters flowing out per minute, which means that there is a net flow of 2L/min. The pool starts with 200 liters of water, so the volume as a function of time is

$$V(t) = 200 + 2t.$$

Using this information, we can write out and solve the differential equation.

$$\frac{dS}{dt} = (\text{rate of substance entering the pool}) - (\text{rate of substance leaving the pool})$$
$$\frac{dS}{dt} = 0 - (\text{rate of substance leaving the pool})$$
$$\frac{dS}{dt} = 0 - (3L/min)(\frac{S(t)}{V(t)})$$
$$\frac{dS}{dt} = 0 - (3L/min)(\frac{S(t)}{200+2t})$$
$$\frac{dS}{dt} = -(\frac{3S(t)}{200+2t}).$$

This is a separable differential equation, so we will solve accordingly.

$$\begin{aligned} \frac{dS}{dt} &= -(\frac{3S(t)}{200+2t}) \\ \frac{1}{S(t)}dS &= \frac{-3}{200+2t}dt \\ \ln|S(t)| &= \frac{-3}{2}\ln|200+2t| + C \\ S(t) &= e^{\frac{-3}{2}\ln|200+2t|+C} \end{aligned}$$

Now, we will use the fact that $S(0) = 500$ to solve for C.

$$\begin{aligned} 500 &= e^{\frac{-3}{2}\ln|200+2(0)|+C} \\ e^C &= 500e^{\frac{3}{2}\ln|200|} \end{aligned}$$

Plugging back in, we find that

$$
\begin{aligned}
S(t) &= e^{\frac{-3}{2}\ln|200+2t|+C} \\
S(t) &= 500e^{\frac{3}{2}\ln|200|}e^{\frac{-3}{2}\ln|200+2t|} \\
S(t) &= 500e^{\frac{3}{2}\ln|200|-\frac{3}{2}\ln|200+2t|} \\
S(t) &= 500e^{\frac{3}{2}\ln|200|-\frac{3}{2}\ln|200+2t|} \\
S(t) &= 500e^{\frac{3}{2}\ln\left|\frac{200}{200+2t}\right|} \\
S(t) &= 500e^{\ln\left|\left(\frac{200}{200+2t}\right)^{3/2}\right|} \\
S(t) &= 500\left(\frac{200}{200+2t}\right)^{3/2}.
\end{aligned}
$$

From here, we need to answer the original question: does the pool become safe before it overflows? Notice that the volume is continuously increasing but the amount of chlorine in the pool is continuously decreasing, which means that the concentration of chlorine in the pool is continuously decreasing as well. Therefore, we can conclude that if the water is safe *when* the pool overflows, then it should have been safe at some point *before* the pool overflowed as well.

Note that the pool overflows when $V(t) = 400$, which is at $t = 100$. At this time, we can see that

$$
\begin{aligned}
S(100) &= 500\left(\frac{200}{400}\right)^{3/2} \\
S(100) &= 500\left(\frac{1}{2}\right)^{3/2} \\
S(100) &= \frac{500}{2^{3/2}} \\
S(100) &< \frac{500}{2} \\
S(100) &< 250.
\end{aligned}
$$

If $S(100) < 250$, then we know that $S(100) < 400$ so the concentration is less than $\frac{250mg}{400L}$ which is less than $1mg/L$. Thus, the concentration of the pool is safe when it overflows, so it must have been safe at some point beforehand too.

(2) Evaluate the integral

$$\int_1^\infty \frac{dx}{x^2(x+2)},$$

or demonstrate that the integral diverges.

We begin by attempting Partial Fraction Decomposition on the integral. We note that we are looking for

$$\frac{1}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}.$$

Multiplying everything by the denominator of the original fraction, we find

$$1 = A(x)(x+2) + B(x+2) + Cx^2,$$

solving for which gives us $A = -\frac{1}{4}$, $B = \frac{1}{2}$, and $C = \frac{1}{4}$. Now, we can integrate the partial fraction to find

$$\begin{aligned}\int_1^\infty \frac{dx}{x^2(x+2)} &= \int_1^\infty \left(-\frac{1}{4}\frac{1}{x} + \frac{1}{2}\frac{1}{x^2} + \frac{1}{4}\frac{1}{x+2}\right) dx \\ &= -\frac{1}{4}\ln|x| - \frac{1}{2x} + \frac{1}{4}\ln|x+2| + C \\ &= -\frac{1}{2x} + \frac{1}{4}\ln\left|\frac{x+2}{x}\right| + C.\end{aligned}$$

Evaluating this along the bounds, we find

$$\begin{aligned}\int_1^\infty \frac{dx}{x^2(x+2)} &= \lim_{a\to\infty}\int_1^a \frac{dx}{x^2(x+2)} \\ \lim_{a\to\infty}\int_1^a \frac{dx}{x^2(x+2)} &= \lim_{a\to\infty} -\frac{1}{2x} + \frac{1}{4}\ln\left|\frac{x+2}{x}\right| \Big|_1^a \\ \lim_{a\to\infty}\int_1^a \frac{dx}{x^2(x+2)} &= \lim_{a\to\infty} -\frac{1}{2a} + \frac{1}{4}\ln\left|\frac{a+2}{a}\right| - \left(-\frac{1}{2(1)} + \frac{1}{4}\ln\left|\frac{(1)+2}{(1)}\right|\right) \\ \lim_{a\to\infty}\int_1^a \frac{dx}{x^2(x+2)} &= -(0) + \frac{1}{4}\ln|1| - \left(-\frac{1}{2} + \frac{1}{4}\ln|3|\right) \\ \lim_{a\to\infty}\int_1^a \frac{dx}{x^2(x+2)} &= \boxed{\frac{1}{2} - \frac{\ln(3)}{4}}\end{aligned}$$

(3) Determine whether each of the following series converges or diverges.

(a) $0.9 - 0.99 + 0.999 - 0.9999 + 0.99999 \cdots$

Note that the limit of this sequence is 1, so by the **Test for Divergence** this **diverges**.

(b) $\frac{1}{2^2} + \frac{2}{3^2} + \frac{3}{4^2} + \cdots$

We can rewrite this series as

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)^2}.$$

From here, we note that it compares nicely to $\sum_{n=1}^{\infty} \frac{1}{n}$. Comparison Test doesn't work because the harmonic series diverges, but by **Limit Comparison Test** we find

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{n}{(n+1)^2}}{\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{n^2}{(n+1)^2} \\ &= \lim_{n\to\infty} \frac{n}{n+1}^{\,2} \\ &= 1, \end{aligned}$$

so by the Limit Comparison Test they both **diverge**.

(c) $\sum_{n=1}^{\infty} \left(\frac{n^2 + 2n}{n^3 + 1} - \frac{1}{2} \right)^n$

Every term in this sequence is taken to the power of n, so we will use the root test.

$$\begin{aligned} \lim_{n\to\infty} \sqrt[n]{|a_n|} &= \sqrt[n]{\left| \left(\frac{n^2 + 2n}{n^3 + 1} - \frac{1}{2} \right)^n \right|} \\ \lim_{n\to\infty} \sqrt[n]{|a_n|} &= \lim_{n\to\infty} \left| \frac{n^2 + 2n}{n^3 + 1} - \frac{1}{2} \right| \\ \lim_{n\to\infty} \sqrt[n]{|a_n|} &= \lim_{n\to\infty} \left| 0 - \frac{1}{2} \right| \\ \lim_{n\to\infty} \sqrt[n]{|a_n|} &= \frac{1}{2} \end{aligned}$$

so by the **Root Test**, because $\frac{1}{2} > 0$, this series **converges**.

(4) Solve the following differential equations.

(a) $xy' - y = x \ln x$

This differential equation is linear, so we will rewrite it as

$$y' - \frac{1}{x}y = \ln x.$$

We can then see that the integrating factor is

$$I(x) = e^{\int -\frac{1}{x}dx} = e^{-\ln(x)dx} = x^{-1},$$

so we multiply our entire differential equation by the integrating factor and then solve accordingly.

$$\begin{aligned}
y' - \frac{1}{x}y &= \ln x \\
I(x)y' - I(x)\frac{1}{x}y &= I(x)\ln x \\
\frac{1}{x}y' - \frac{1}{x^2}y &= \frac{1}{x}\ln x \\
\left(\frac{1}{x}y\right)' &= \frac{\ln(x)}{x} \\
\frac{1}{x}y &= \int \frac{\ln(x)}{x}dx \\
\frac{1}{x}y &= \int udu[u = \ln(x)] \\
\frac{1}{x}y &= \frac{1}{2}u^2 + C \\
\frac{1}{x}y &= \frac{1}{2}\ln(x)^2 + C \\
y &= \boxed{\frac{1}{2}x\ln(x)^2 + Cx}
\end{aligned}$$

(b) $y' = 0.1y - 0.00005y^2, y(0) = 500,$

Let's change the variables for a second. If we choose different variables, we can see that it is equal to

$$\frac{dP}{dt} = 0.1P - 0.00005P^2,$$

which can be rewritten as

$$\frac{dP}{dt} = 0.1P(1 - \frac{1}{2000}P),$$

which is a logistic equation! Thus, the final solution is

$$y(x) = \frac{2000}{1 + \frac{2000-500}{500}e^{-0.01x}} = \boxed{\frac{2000}{1 + 3e^{-0.01x}}}$$

(5) Find the Taylor series of the following functions:

$$(a)\ f(x) = \frac{1}{\sqrt{x}}, a = 1$$

Note that

$$\frac{1}{\sqrt{x}} = \frac{1}{\sqrt{1+(x-1)}} = (1+(x-1))^{-1/2},$$

so the Taylor series is

$$\boxed{\sum_{n=0}^{\infty} \binom{-1/2}{n} (x-1)^n}$$

$$(b)\ f(x) = 1 - \sin^2(x), a = 0$$

We can use the identity

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x)).$$

We know that the Taylor series for $\cos(x)$ is

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!},$$

so the Taylor series for $\cos(2x)$ is

$$\cos(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} x^{2n}}{(2n)!},$$

meaning that

$$1 - \cos(2x) = 1 - \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} x^{2n}}{(2n)!}$$

$$1 - \cos(2x) = 1 - \left(1 + \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n} x^{2n}}{(2n)!}\right)$$

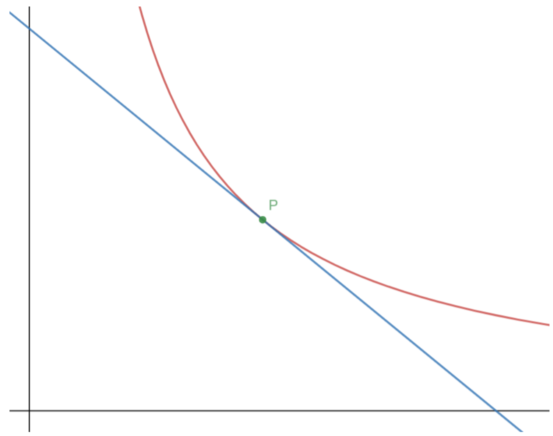
$$1 - \cos(2x) = -\sum_{n=1}^{\infty} (-1)^n \frac{2^{2n} x^{2n}}{(2n)!}$$

$$1 - \cos(2x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^{2n} x^{2n}}{(2n)!}$$

$$\frac{1}{2}(1 - \cos(2x)) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^{2n-1} x^{2n}}{(2n)!}$$

$$1 - \left(\frac{1}{2}(1 - \cos(2x))\right) = \boxed{1 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^{2n-1} x^{2n}}{(2n)!}}$$

(6) Find the curve that passes through the point $(3,2)$ and has the property that if the tangent line is drawn at any point P on the curve, then the part of the tangent line that lies in the first quadrant is bisected by P. In the diagram below, you can see what it should look like: the two parts of the line separated by the point P are of equal length in the first quadrant. (Hint: first find the slope of the line through the point $(3,2)$ whose part in the first quadrant is bisected by $(3,2)$. How can we generalize this process?)



Let $P(a,b)$ be any first-quadrant point on the curve $y = f(x)$. The tangent line at P has equation $y - b = f'(a)(x - a)$, which can be rewritten as $y - b = mx - ma$, where $m = f'(a)$. The x-intercept is when $y = 0$, which is at

$$\begin{aligned} 0 - b &= mx - ma \\ -b &= m(x-a) \\ -\frac{b}{m} &= x - a \\ x &= a - \frac{b}{m}. \end{aligned}$$

Meanwhile, the x-intercept is at

$$\begin{aligned} y - b &= m(0) - ma \\ y &= b - ma. \end{aligned}$$

Since the tangent line is bisected by P, we can see that the distance from the x-axis to P is equal to the distance from the y-axis to P. Writing this out as an equation, we find

$$\begin{aligned} D(x_{int}, P) &= D(y_{int}, P) \\ \sqrt{[a-(a-b/m)]^2 + (b-0)^2} &= \sqrt{(a-0)^2 + [b-(b-am)]^2} \\ [a-(a-b/m)]^2 + (b-0)^2 &= (a-0)^2 + [b-(b-am)]^2 \\ (b/m)^2 + b^2 &= a^2 + (am)^2 \\ b^2 + m^2 b^2 &= m^2 a^2 + a^2 m^4 \\ a^2 m^4 + (a^2 - b^2)m^2 - b^2 &= 0 \\ (a^2 m^2 - b^2)(m^2 + 1) &= 0 \end{aligned}$$

The factor on the right can not equal 0, so we must have $a^2m^2 - b^2 = 0$ or $m^2 = b^2/a^2$. We know that m must turn out to be negative due to the fact that it intercepts both the positive x and positive y axes, so we can simplify this to $m = -b/a$. We can rewrite this as a differential equation and solve.

$$\begin{aligned} m &= -b/a \\ \frac{dy}{dx} &= -\frac{y}{x} \\ \frac{dy}{y} &= -\frac{dx}{x} \\ \int \frac{dy}{y} &= -\int \frac{dx}{x} \\ \ln(y) &= -\ln(x) + C \\ y &= e^{-\ln(x)+C} \\ y &= x^{-1}e^{C}. \end{aligned}$$

We know that the point $(3, 2)$ is on the curve, so we find

$$\begin{aligned} 2 &= \frac{1}{3}e^{C} \\ 6 &= e^{C}, \end{aligned}$$

which leaves us with the final equation

$$\boxed{y = \frac{6}{x}}$$