MATH 1B DISCUSSION WORKSHEET - 11/27/18

LINEAR HOMOGENEOUS SECOND ORDER DIFFERENTIAL EQUATIONS, AND METHOD OF UNDETERMINED COEFFICIENTS ANSWERS

1. Homogeneous Equations

Solving these problems boils down to three cases. Fill in the following table to help you remember how to solve each of the three.

Solutions to the differential equation $ay'' + by' + c = 0$

Types of Roots	Value of $b^2 - 4ac$	General Solution
r_1 and r_2 are real and distinct	positive	$y = c_1e^{r_1x} + c_2e^{r_2x}$
r_1 is a double root	zero	$y = c_1e^{rx} + c_2xe^{rx}$
r_1 and r_2 are complex conjugates $\alpha \pm \beta i$	negative	$y = e^{\alpha x}(C_1\cos(\beta x) + C_2\sin(\beta x))$

You would think that by now I should know how to center text in a table, but I couldn't figure it out in a minute's worth of google searches so I gave up on it. Pretend they're centered.

Using the table, classify the following differential equations into one of the three cases, and use that classification to determine the general solution. Using this and the values given, solve for the values of the constants.

(1) $y'' + 6y' + 5y = 0, y(0) = 1, y'(0) = 0$

Our auxiliary equation is r^2+6r+5, which has $b^2-4ac = 16$ and belongs in the first row of the table. Its roots are $r = -1, r = -5$, so its general solution is $y = c_1e^{-x} + c_2e^{-5x}$. Solving for c_1 and c_2 with the initial values, we find

$$y(0) = c_1e^{-(0)} + c_2e^{-5(0)} = c_1 + c_2 = 1$$
$$y'(0) = -c_1e^{-(0)} + -5c_2e^{-5(0)} = -c_1 - 5c_2 = 0.$$

Solving, we find $c_1 = \frac{5}{4}$ and $c_2 = -\frac{1}{4}$, so our final solution is

$$\boxed{y = \frac{5}{4}e^{-x} - \frac{1}{4}e^{-5x}}$$

(2) $y'' + 12y' + 37y = 0, y(0) = 4, y'(0) = 0$

Our auxiliary equation is $r^2 + 12r + 37$, which has $b^2 - 4ac = -100$ and belongs in the third row of the table. Its roots are $r = -6 \pm i$, so its general solution is $y = e^{-6x}(C_1\cos(x) + C_2\sin(x))$. Solving for c_1 and c_2 with the initial values, we find

$$y(0) = e^{-6(0)}(C_1\cos(0) + C_2\sin(0)) = c_1 = 4$$
$$y(0) = -6e^{-6(0)}(C_1\cos(0) + C_2\sin(0)) + e^{-6x}(-C_1\sin(0) + C_2\cos(0)) = -6c_1 - c_2 = 0.$$

Solving, we find $C_1 = 4$ and $C_2 = -24$, so our final solution is

$$\boxed{y = e^{-6x}(4\cos(x) - 24\sin(x))}$$

(3) $y'' - 8y' + 16y = 0, y(0) = -3, y'(0) = 4$

Our auxiliary equation is $r^2 - 8r + 16$, which has $b^2 - 4ac = 0$ and belongs in the second row of the table. Its roots are $r = 4, r = 4$, so its general solution is $y = c_1 e^{4x} + c_2 x e^{4x}$. Solving for c_1 and c_2 with the initial values, we find

$$y(0) = c_1 e^{4(0)} + c_2(0)e^{4(0)} = c_1 = -3$$
$$y'(0) = 4c_1 e^{4(0)} + 4c_2(0)e^{4(0)} + c_2 e^{4(0)} = 4c_1 + c_2 = 4.$$

Solving, we find $c_1 = -3$ and $c_2 = 16$, so our final solution is

$$\boxed{y = -3e^{4x} + 4xe^{4x}}$$

2. Nonhomogeneous equations part 1: Method of Undetermined Coefficients

For each of the following differential equations, what guess should I be using for my particular solution?

(1) $y'' + 6y' + 4y = e^t$

The number 1 is not a root to the auxiliary equation $r^2 + 6r + 4 = 0$, which means that we can simply guess $\boxed{Ae^t}$.

(2) $y'' + 2y' + y = t^3$

We simply guess a polynomial of the same degree as the polynomial on the right, so we guess $\boxed{At^3 + Bt^2 + Ct + D}$.

(3) $y'' + 3y' + 2y = 2\sin(t) + \cos(t)$

The left hand side has real roots so its homogeneous equation will not contain any sines or cosines. Thus, it will not overlap with our guess. We can guess $\boxed{A\sin(t) + B\cos(t)}$.

(4) $y'' - 6y' + 9y = e^{3t}$

The homogeneous solution to this differential equation is $y_h = c_1 e^{3t} + c_2 t e^{3t}$. Because the right hand side is $e^3 t$, our first guess would be $y = Ae^{3t}$. However, because this is contained in the homogeneous equation, we will try multiplying it by t to guess $y = Ate^{3t}$. However, this is still contained in the homogeneous equation! We must multiply by t again, to find that our final guess is $\boxed{y = At^2 e^{3t}}$.

(5) $y'' + 16y = \cos(4t)$

The auxiliary equation related to this differential equation is $r^2 + 4$, which has roots $\pm 4i$ and gives homogeneous solutions $y_h = c_1\cos(4t) + c_2\sin(4t)$. This is rather unfortunate, because our guess would have been $A\cos(4t) + B\sin(4t)$. In this case, we should simply multiply our guess by t, so our final guess is $\boxed{y_p = At\cos(4t) + Bt\sin(4t)}$.

(6) $y'' + 2y' + y = e^x\cos(x)$

In this situation, note that the auxiliary equation is $r^2 + 2r + 1$, so the homogeneous solution is $y_h = c_1 e^{-x} + c_2 x e^{-x}$. In this case, our guess would not contain e^{-x}, so we don't need to worry about any overlaps. As a result, our guess would be $\boxed{Ae^x\cos(x) + Be^x\sin(x)}$.