SECTION 17.4 SOLUTIONS

1. Preliminary Series Manipulations That You Should Know

If we want to change the indexing or the starting terms of any series, we have a few tools at our disposal to help us do so. You should take some time for yourself to understand why each of them work.

(a) When we want to increase the exponent on x: Replacing every term n with $n+1$, and decreasing the index by 1.

i) $\displaystyle\sum_{n=1}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{n+1} x^{n+1}$

ii) $\displaystyle\sum_{n=3}^{\infty} (n+2)c_n x^{2n} = \sum_{n=2}^{\infty} ((n+1)+2)c_{n+1} x^{2(n+1)} = \sum_{n=2}^{\infty} (n+3)c_{n+1} x^{2n+2}$

(b) When we want to decrease the exponent on x: Replacing every term n with $n-1$, and increasing the index by 1.

i) $\displaystyle\sum_{n=0}^{\infty} c_n x^n = \sum_{n=1}^{\infty} c_{n-1} x^{n-1}$

(c) When we want to increase the starting term, but don't want to affect the inside: Pulling the first term(s) out of the series.

i) $\displaystyle\sum_{n=0}^{\infty} c_n x^n = c_0 x^0 + \sum_{n=1}^{\infty} c_n x^n$

ii) $\displaystyle\sum_{n=2}^{\infty} \frac{f^{(n)}(x)}{n!} x^n = \frac{f^{(2)}(x)}{2!} x^2 + \frac{f^{(3)}(x)}{3!} x^3 + \sum_{n=4}^{\infty} \frac{f^{(n)}(x)}{n!} x^n$

(d) When we want to decrease the starting term, but don't want to affect the inside: decreasing the starting term means that our sum has extra terms, so we must subtract them from the new sequence.

i) $\displaystyle\sum_{n=1}^{\infty} c_n x^n = -c_0 x^0 + \sum_{n=0}^{\infty} c_n x^n$

2. How to Solve these Problems

We will use a seven-step procedure to solve these problems.

(i) Rewrite y and the differential equation using

$$y = \sum_{n=0}^{\infty} c_n x^n,\ y' = \sum_{n=0}^{\infty} (n+1)c_{n+1} x^n,\ y'' = \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n.$$

(ii) Change the series to make all series have an x^n term using methods (a) and (b) above.
(iii) Rewrite the series to give all of them the same starting index using methods (c) and (d).
(iv) Combine the series into one large series.
(v) Use this large series to find a recursive relationship between constants.
(vi) Use this recursive relationship and some plugging in to find an explicit formula for c_n.
(vii) Rewrite y in terms of c_n. If possible, identify the function that this y's Taylor series corresponds to, and rewrite y as such.