MATH 1B DISCUSSION WORKSHEET - 8/28/18

TRIGONOMETRIC INTEGRALS AND SUBSTITUTIONS ANSWERS

1. Previously covered trigonometric integrals

$$\int \sin^{2018}(x)\cos(x)dx$$

Because we have a single cosine term, we proceed with u-substitution.

$$u = \sin(x)$$
$$du = \cos(x)\,dx$$

This changes the integral into

$$\int u^{2018}\,du = \frac{1}{2019}u^{2019},$$

so our final answer is $\boxed{\frac{1}{2019}\sin^{2019}(x) + C}$.

$$\int \sin^4(x)\,dx$$

One way to solve this would be with the reduction formula, or by doing integration by parts with $u = \sin^3(x)$ and $dv = \sin(x)dx$). However, our best bet here is to use the double-angle identities to try to get rid of the exponent. Knowing that

$$\cos(2\theta) = 1 - 2\sin^2(\theta),$$

we find

$$\sin^2(\theta) = \frac{1}{2}(1 - \cos(2\theta))$$

so the integral can be rewritten as

$$\begin{aligned}
\int \sin^4(x)\,dx &= \int \left(\sin^2(x)\right)^2\,dx \\
&= \int \left(\frac{1}{2}(1-\cos(2x))\right)^2\,dx \\
&= \frac{1}{4}\int (1-\cos(2x))^2\,dx \\
&= \frac{1}{4}\int 1 - 2\cos(2x) + \cos^2(2x)\,dx
\end{aligned}$$

From here, we'll have to use the other double-angle identity

$$\cos(2\theta) = 2\cos^2(\theta) - 1,$$

which gives us

$$\cos^2(\theta) = \frac{1}{2}(1 + \cos(2\theta)).$$

This leads us to

$$\begin{aligned}
\int \sin^4(x) &= \frac{1}{4}\int 1 - 2\cos(2x) + \cos^2(2x)\,dx \\
&= \frac{1}{4}\int 1 - 2\cos(2x) + \frac{1}{2}(1 + \cos(4x))\,dx \\
&= \frac{1}{4}\left(x - \sin(2x) + \frac{1}{2}x + \frac{1}{8}\sin(4x)\right) + C \\
&= \boxed{\frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C}.
\end{aligned}$$

2. Powers of trigonometric functions

$$\int \sin^9(x)\cos^3(x)\,dx$$

We recognize that in this integral, both of these terms are to odd powers. In this case, we should choose the lower power to split up, which is $\cos^3(x)$. Noting that $\cos^3(x) = \cos^2(x)\cos(x) = (1 - \sin^2(x))\cos(x)$, we find that

$$\int \sin^9(x)\cos^3(x)\,dx = \int \sin^9(x)(1 - \sin^2(x))\cos(x)\,dx.$$

From here, as the cosine term is to the power of 1, we can simply do u-substitution:

$$\begin{aligned}
u &= \sin(x) \\
du &= \cos(x)\,dx
\end{aligned}$$

From here, we can solve.

$$\begin{aligned}
\int \sin^9(x)\cos^3(x)\,dx &= \int \sin^9(x)(1 - \sin^2(x))\cos(x)\,dx \\
&= \int u^9(1 - u^2)\,du \\
&= \int u^9 - u^{11}\,du \\
&= \frac{1}{10}u^{10} - \frac{1}{12}u^{12} + C \\
&= \boxed{\frac{1}{10}\sin^{10}(x) - \frac{1}{12}\sin^{12}(x) + C}.
\end{aligned}$$

$$\int \sin^8(x)\cos^4(x)\,dx$$

(Sorry, this problem was much more tedious than expected. I wouldn't recommend doing it.)

For this problem, we unfortunately see that both powers are even. In this case, we can not rely on the same technique, and must instead use double-angle formulas to rewrite the integral. We found that

$$\sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta))$$

and

$$\cos^2(\theta) = \frac{1}{2}(1+\cos(2\theta)),$$

so we can rewrite the integral:

$$\begin{aligned}
\int \sin^8(x)\cos^4(x)\,dx &= \int \left(\sin^2(x)\right)^4\left(\cos^2(x)\right)^2\,dx \\
&= \int \left(\frac{1}{2}(1-\cos(2x))\right)^4\left(\frac{1}{2}(1+\cos(2x))\right)^2\,dx \\
&= \frac{1}{64}\int (1-\cos(2x))^4(1+\cos(2x))^2\,dx \\
&= \frac{1}{64}\int (1-\cos(2x))^2(1-\cos(2x))^2(1+\cos(2x))^2\,dx \\
&= \frac{1}{64}\int (1-\cos(2x))^2(1-\cos^2(2x))^2\,dx \\
&= \frac{1}{64}\int ((1-\cos(2x)-\cos^2(2x)+\cos^3(2x))^2\,dx \\
&= \frac{1}{64}\int 1-2\cos(2x)-\cos^2(2x)+4\cos^3(2x)-\cos^4(2x)-2\cos^5(2x)+\cos^6(2x)\,dx
\end{aligned}$$

This is a pretty gross integral, but at least we have tools to solve each term individually:

$$\int 1\,dx = x$$

$$\int -2\cos(2x)\,dx = -\sin(2x)$$

$$\int 4\cos^3(2x)\,dx = \int 4\cos(2x)(1-\sin^2(2x))\,dx \to [\text{u} = \sin(2x)] \to \frac{3}{2}\sin(2x) + \frac{1}{6}\sin(6x)$$

$$\int -\cos^4(2x) = [\text{see second problem}] \to \frac{-3}{8}x - \frac{1}{8}\sin(4x) - \frac{1}{64}\sin(8x)$$

$$\int -2\cos^5(2x)\,dx = \int 4\cos(2x)(1-\sin^2(2x))^2 \to [\text{u} = \sin(2x)] \to \frac{-5}{8}\sin(2x) - \frac{5}{48}\sin(6x) - \frac{1}{80}\sin(10x)$$

$$\int \cos^6(2x)\,dx = [\text{use double-angle, then leverage above integrals to skip integrations}]$$

$$= \frac{5}{16}x + \frac{15}{128}\sin(4x) + \frac{3}{128}\sin(8x) + \frac{1}{384}\sin(12x)$$

When we add these all together, we obtain the final answer:

$$\boxed{\frac{7}{1024}x - \frac{1}{512}\sin 2x - \frac{17}{8192}\sin(4x) + \frac{1}{1024}\sin(6x) + \frac{1}{8192}\sin(8x) - \frac{1}{5120}\sin(10x) + \frac{1}{24576}\sin(12x) + C}.$$

(My bad.)

$$\int \tan^9(x)\sec^5(x)\,dx$$

For this problem, as shown in class, we can see that the power of tangent is odd so we can take a factor of $\sec(x)\tan(x)$ out of the product to find a simple u-substitution. Knowing that

$$u = \sec(x)$$
$$du = \sec(x)\tan(x)dx,$$

we find

$$\begin{aligned}
\int \tan^9(x)\sec^5(x)\,dx &= \int \tan^8(x)\sec^4(x)\tan(x)\sec(x)\,dx \\
&= \int (\sec^2(x)-1)^4\sec^4(x)\tan(x)\sec(x)\,dx \\
&= \int \left(u^2-1\right)^4 u^4\,du \\
&= \int u^{12} - 4u^{10} + 6u^8 - 4u^6 + u^4\,du \\
&= \frac{1}{13}u^{13} - \frac{4}{11}u^{11} + \frac{2}{3}u^9 - \frac{4}{7}u^7 + \frac{1}{5}u^5 + C \\
&= \boxed{\frac{1}{13}\sec^{13}(x) - \frac{4}{11}\sec^{11}(x) + \frac{2}{3}\sec^9(x) - \frac{4}{7}\sec^7(x) + \frac{1}{5}\sec^5(x) + C}
\end{aligned}$$

$$\int \tan^8(x)\sec^4(x)\,dx$$

For this problem, we can see that the power of secant is even so we can take out a factor of $\sec^2(x)$ to perform a u-substitution. Knowing that

$$u = \tan(x)$$
$$du = \sec^2(x)dx,$$

we find

$$\begin{aligned}
\int \tan^8(x)\sec^4(x)\,dx &= \int \tan^8(x)\sec^2(x)\sec^2(x)\,dx \\
&= \int \tan^8(x)(\tan^2(x)+1)\sec^2(x)\,dx \\
&= \int u^8(u^2+1)\,du \\
&= \frac{1}{11}u^{11} + \frac{1}{9}u^9 + C \\
&= \boxed{\frac{1}{11}\tan^{11}(x) + \frac{1}{9}\tan^9(x) + C}
\end{aligned}$$

3. Constants within the trigonometric functions

$$\int \sin(3x)\sin(5x)\,dx$$

We can simply use the identity

$$\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

to rewrite the integral. From there, we separate and integrate normally.

$$\int \sin(3x)\sin(5x)\,dx = \frac{1}{2}\int \cos(2x) - \cos(8x)\,dx$$
$$= \boxed{\frac{1}{4}\sin 2x - \frac{1}{16}\sin 8x + C}$$

$$\int \sin(3x)\cos(5x)\,dx$$

We can simply use the identity

$$\sin(A)\cos(B) = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$$

to rewrite the integral. From there, we separate and integrate normally.

$$\int \sin(3x)\cos(5x)\,dx = \frac{1}{2}\int \sin(2x) + \sin(8x)\,dx$$
$$= \boxed{-\frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x + C}$$

$$\int \cos(3x)\cos(5x)\,dx$$

We can simply use the identity

$$\cos(A)\cos(B) = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$$

to rewrite the integral. From there, we separate and integrate normally.

$$\int \cos(3x)\cos(5x)\,dx = \frac{1}{2}\int \cos(2x) + \cos(8x)\,dx$$
$$= \boxed{\frac{1}{4}\sin 2x + \frac{1}{16}\sin 8x + C}$$

4. Trigonometric Substitution

$$\int \frac{\sqrt{x^2-1}}{x^4}\,dx$$

The $\sqrt{x^2-1}$ term in this problem indicates that it's probably a trigonometric substitution problem. Looking at our chart of substitutions, we can see that x^2-1 falls into the category of x^2-a^2 with $a=1$, so we know to perform a substitution with

$$x = \sec(\theta)$$
$$dx = \sec(\theta)\tan(\theta)\,d\theta$$

Substituting these in, we find

$$\begin{aligned}\int \frac{\sqrt{x^2-1}}{x^4}\,dx &= \int \frac{\sqrt{\sec^2(\theta)-1}}{\sec^4(\theta)}(\sec(\theta)\tan(\theta)\,d\theta)\\ &= \int \frac{\sec(\theta)\tan(\theta)\sqrt{\tan^2(\theta)}}{\sec^4(\theta)}\,d\theta\\ &= \int \frac{\tan^2(\theta)}{\sec^3(\theta)}\,d\theta\end{aligned}$$

At this point, we could multiply this expression by some number of sec terms or tan terms to allow us to do u-substitution with each of those terms, but for this problem we're better off just simplifying the integrand. Upon simplification, we find

$$\begin{aligned}\int \frac{\tan^2(\theta)}{\sec^3(\theta)}\,d\theta &= \int \frac{\left(\frac{sin^2(\theta)}{\cos^2(\theta)}\right)}{\left(\frac{1}{\cos^3(\theta)}\right)}\,d\theta\\ &= \int \sin^2(\theta)\cos(\theta)\,d\theta\end{aligned}$$

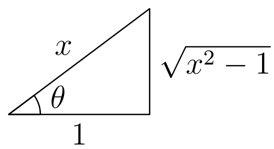
From here, we can simply do u-substitution to integrate because the cosine term doesn't have an exponent. With $u=\sin(\theta)$ and $du=\cos(\theta)\,d\theta$, we find

$$\int \sin^2(\theta)\cos(\theta)\,d\theta = \int u^2\,du = \frac{1}{3}u^3 + C$$

Now, all we have left to do is un-substitute our variables (remember, we did two substitutions!). First, we used the substitution $u=\sin(\theta)$, so we can rewrite

$$\frac{1}{3}u^3 + C = \frac{1}{3}\sin^3(\theta) + C.$$

All that's left is to find a way to rewrite $\sin(\theta)$ in terms of x. Our original substitution was $x=\sec(\theta)$, where $\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{1}{Adj/Hyp} = \frac{Hyp}{Adj} = x$, so we deduce that our right triangle looks like this:



With a triangle like this, we can see that $\sin(\theta) = \frac{\sqrt{x^2-1}}{x}$. Therefore, our final answer is

$$\boxed{\frac{1}{3}\left(\frac{\sqrt{x^2-1}}{x}\right)^3 + C}$$

$$\int \frac{dx}{\sqrt{x-x^2}}$$

We initially want to use trigonometric substitution for this problem, but unfortunately the expression in the square root doesn't look as we'd want it to; there's no constant. However, we also know that if we complete the square for this polynomial, we can find some combination of a square number and a constant, which is good enough for us. Completing the square for $x - x^2$ gives

$$\begin{aligned} x - x^2 &= -(x^2 - x) \\ &= -\left(x^2 - x + \frac{1}{4} - \frac{1}{4}\right) \\ &= -\left(\left(x - \frac{1}{2}\right)^2 - \frac{1}{4}\right) \\ &= \frac{1}{4} - \left(x - \frac{1}{2}\right)^2, \end{aligned}$$

which we can use! Rewriting our integral gives

$$\int \frac{dx}{\sqrt{x-x^2}} = \int \frac{dx}{\sqrt{\frac{1}{4} - \left(x - \frac{1}{2}\right)^2}}.$$

From here, we return it to the notation we know by performing a u-sub with $u = x - \frac{1}{2}$, giving

$$\int \frac{du}{\sqrt{\frac{1}{4} - u^2}}.$$

Now, the term under the square root is of the form $a^2 - x^2$, so we can use the substitution $u = a\sin(\theta)$ and $du = a\cos(\theta)\,d\theta$ with $a^2 = \frac{1}{4}$ so $a = \frac{1}{2}$ and this gives us

$$\begin{aligned} \int \frac{du}{\sqrt{\frac{1}{4} - u^2}} &= \int \frac{\frac{1}{2}\cos(\theta)\,d\theta}{\sqrt{\frac{1}{4} - \frac{1}{4}\sin^2(\theta)}} \\ &= \int \frac{\frac{1}{2}\cos(\theta)\,d\theta}{\sqrt{\frac{1}{4}(1 - \sin^2(\theta))}} \\ &= \int \frac{\frac{1}{2}\cos(\theta)\,d\theta}{\sqrt{\frac{1}{4}\cos^2(\theta)}} \\ &= \int \frac{\frac{1}{2}\cos(\theta)\,d\theta}{\frac{1}{2}\cos(\theta)} \\ &= \int d\theta \\ &= \theta. \end{aligned}$$

Now, all that's left to do is substitute the variables back. First, we know that $u = \frac{1}{2}\sin(\theta)$, which gives us $\theta = \sin^{-1}(2u)$. Then, we know that $u = x - \frac{1}{2}$, so by undoing all of our substitutions, our final answer is

$$\boxed{\sin^{-1}(2(x - \frac{1}{2})) = \sin^{-1}(2x - 1)}.$$

$$\int \frac{(x+3)^5}{(40-6x-x^2)^{3/2}}\,dx$$

As in the previous problem, the term that we would like to use to do trigonometric substitution is not in a suitable form for us to do so. However, by completing the square, we recognize that

$$\begin{aligned} 40-6x-x^2 &= 40-(x^2+6x) \\ &= 40-(x^2+6x+9-9) \\ &= 40-((x+3)^2-9) \\ &= 49-(x+3)^2. \end{aligned}$$

Now it's in a form we can use again! Rewriting our original integral gives us

$$\int \frac{(x+3)^5}{(49-(x+3)^2)^{3/2}}\,dx,$$

and we can see that a u-sub of $u = x+3$ would be useful. Doing so, we find

$$\int \frac{(x+3)^5}{(49-(x+3)^2)^{3/2}}\,dx = \int \frac{u^5}{(49-u^2)^{3/2}}\,du.$$

From here, we'll use our trigonometric substitution. Seeing that the term in the exponent is of the form $a^2 - x^2$ with $a = 7$, we use the substitution $u = 7\sin(\theta)$ with $du = 7\cos(\theta)\,d\theta$ and simplify:

$$\begin{aligned} \int \frac{u^5}{(49-u^2)^{3/2}}\,du &= \int \frac{(7\sin(\theta))^5}{(49-(7\sin(\theta))^2)^{3/2}} 7\cos(\theta)\,d\theta \\ &= \int \frac{7^6\sin^5(\theta)\cos(\theta)}{(49-49\sin^2(\theta))^{3/2}}\,d\theta \\ &= \int \frac{7^6\sin^5(\theta)\cos(\theta)}{(49\cos^2(\theta))^{3/2}}\,d\theta \\ &= \int \frac{7^6\sin^5(\theta)\cos(\theta)}{7^3\cos^3(\theta)}\,d\theta \\ &= 343\int \frac{\sin^5(\theta)}{\cos^2(\theta)}\,d\theta \end{aligned}$$

While this integral is unlike integrals we've seen in class, the fact that the every term is in the form sin or cos and that the cos term is in the denominator suggests a u-sub with $v = \cos(\theta)$ and $dv = -\sin(\theta)d\theta$ (I don't want to repeat the use of the variable u). If we try to find one, we end up with

$$343\int \frac{\sin^5(\theta)}{\cos^2(\theta)}\,d\theta = -343\int \frac{\sin^4(\theta)}{\cos^2(\theta)}(-\sin(\theta)\,d\theta).$$

Now that the sin term is of even degree, we can rewrite it in terms of cos, giving us

$$-343\int \frac{\sin^4(\theta)}{\cos^2(\theta)}(-\sin(\theta)\,d\theta) = -343\int \frac{\left(1-\cos^2(\theta)\right)^2}{\cos^2(\theta)}(-\sin(\theta)\,d\theta),$$

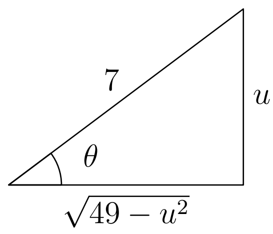
so performing the u-substitution leads us to

$$\begin{aligned}-343\int \frac{\left(1-\cos^2(\theta)\right)^2}{\cos^2(\theta)}(-\sin(\theta)\,d\theta) &= -343\int \frac{\left(1-v^2\right)^2}{v^2}\,dv \\ &= -343\int \frac{1-2v^2+v^4}{v^2}\,dv \\ &= -343\int \frac{1}{v^2}-2+v^2\,dv \\ &= -343\left[-\frac{1}{v}-2v+\frac{1}{3}v^3\right]\end{aligned}$$

Finally, we can undo our substitutions. We had $v=\cos(\theta)$, so we find

$$-343\left[-\frac{1}{v}-2v+\frac{1}{3}v^3\right] = -343\left[-\frac{1}{\cos(\theta)}-2\cos(\theta)+\frac{1}{3}\cos^3(\theta)\right].$$

Then, we have $u=7\sin(\theta)$ so $\frac{u}{7}=\sin(\theta)$, which suggests the triangle



so $\cos(\theta)=\frac{\sqrt{49-u^2}}{7}$. Thus, we can make the substitution

$$-343\left[-\frac{1}{\cos(\theta)}-2\cos(\theta)+\frac{1}{3}\cos^3(\theta)\right] = -343\left[-\frac{7}{\sqrt{49-u^2}}-\frac{2}{7}\sqrt{49-u^2}+\frac{1}{3}\left(\frac{\sqrt{49-u^2}}{7}\right)^3\right].$$

Finally, we know that $u=x+3$, so our final answer is

$$\boxed{-343\left[-\frac{7}{\sqrt{49-(x+3)^2}}-\frac{2}{7}\sqrt{49-(x+3)^2}+\frac{1}{3}\left(\frac{\sqrt{49-(x+3)^2}}{7}\right)^3\right]}$$

.