MATH 1B DISCUSSION WORKSHEET - 9/5/18

PARTIAL FRACTION DECOMPOSITION + INTEGRAL REVIEW ANSWERS

1. Partial Fraction Decomposition

$$\int \frac{x^2}{x^2+1}dx$$

We can rewrite the integral as

$$\int \frac{x^2}{x^2+1}dx = \int \frac{x^2+1-1}{x^2+1}dx = \int \frac{x^2+1}{x^2+1} - \frac{1}{x^2+1}dx = \int \left(1 - \frac{1}{x^2+1}\right) dx.$$

(We could have also done long division.)

From here, we know that the left side can be integrated easily, and the right side is the derivative of $\tan^{-1}(x)$. Therefore, the final answer is

$$\int \left(1 - \frac{1}{x^2+1}\right) dx = \boxed{x - \tan^{-1}(x) + C}.$$

$$\int \frac{2x^3-1}{x^4+x}dx$$

To do our partial fraction decomposition, we first note that the numerator has lower degree than the denominator. This tells us that we should begin normally decomposing the fraction by first factoring the denominator.

$$x^4 + x = x(x^3+1) = x(x+1)(x^2-x+1).$$

From here, we know that we are trying to find values of A, B, C, and D such that

$$\frac{A}{x} + \frac{B}{x+1} + \frac{Cx+D}{x^2-x+1} = \frac{2x^3-1}{x^4+x}.$$

Multiplying the right hand side's denominator to both sides of the equation, we get

$$(A)(x^3+1) + (B)(x^3-x^2+x) + (Cx+D)(x^2+x) = 2x^3-1.$$

Expanding the rest of the equation out, we find

$$\begin{aligned}(A)(x^3+1) + (B)(x^3-x^2+x) + (Cx+D)(x^2+x) &= 2x^3-1\\ Ax^3 + A + Bx^3 - Bx^2 + Bx + Cx^3 + Dx^2 + Cx^2 + Dx &= 2x^3+1\\ (A+B+C)x^3 + (-B+C+D)x^2 + (B+D)x + A &= 2x^3+1\end{aligned}$$

This tells us that $A = 1$, which we can use to write

$$\begin{aligned}1 + B + C &= 2\\ -B + C + D &= 0\\ B + D &= 0.\end{aligned}$$

Solving this system (however we want) gives us

$$B = \frac{1}{3}, C = \frac{2}{3}, D = \frac{-1}{3},$$

so our final integral has the form

$$\int \left(\frac{1}{x} + \frac{1}{3(x+1)} + \frac{2x-1}{3(x^2-x+1)} \right) dx.$$

The two integrals on the left integrate to $\ln|x|$ and $\frac{1}{3}\ln|x+1|$, respectively, and we can use u-substitution for the final one. Using $u = x^2 - x + 1$ and $du = 2x - 1dx$, we find

$$\begin{aligned}\int \frac{2x-1}{3(x^2-x+1)} dx &= \frac{1}{3}\int \frac{2x-1}{x^2-x+1} dx \\ &= \frac{1}{3}\int \frac{du}{u} \\ &= \frac{1}{3}\ln|x^2-2x+1|,\end{aligned}$$

so our final answer is

$$\ln|x| + \frac{1}{3}\ln|x+1| + \frac{1}{3}\ln|x^2-x+1| + C = \boxed{\ln|x| + \frac{1}{3}\ln|x^3+1| + C}.$$

$$\int \frac{dx}{e^x+1}$$

Because we don't have a great approach to this problem, we will go for broke and try to u-sub. Using $u = e^x$, we find $du = e^x dx$, which tells us $\frac{1}{e^x}du = dx$ so we can write $\frac{1}{u}du = dx$. Thus, we can rewrite the integral and use partial fraction decomposition to find:

$$\begin{aligned}\int \frac{dx}{e^x+1} &= \int \frac{1/udu}{u+1} \\ &= \int \frac{1}{(u)(u+1)} du \\ &= \int \left(\frac{1}{u} - \frac{1}{u+1} \right) du \\ &= \ln|u| - \ln|u+1| + C \\ &= \ln|e^x| - \ln|e^x+1| + C \\ &= \boxed{|x| - \ln|e^x+1| + C}.\end{aligned}$$

2. One Multi-step Problem

[From *Calculus* by David Patrick.]

$$\int \frac{3}{x^3-1}dx$$

We begin with Partial Fraction Decomposition. We know that $x^3-1=(x-1)(x^2+x+1)$, so we know that we are looking for A, B, and C such that

$$\frac{3}{x^3-1}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}.$$

Multiplying the common denominator gives us

$$\begin{aligned}\frac{3}{x^3-1}&=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}\\ 3&=A(x^2+x+1)+(Bx+C)(x-1)\\ 3&=Ax^2+Ax+A+Bx^2+Cx-Bx-C\\ 3&=(A+B)x^2+(A-B+C)x+(A-C)\end{aligned}$$

Solving for A, B, C, gives us $A=1$, $B=-1$, $C=-2$.

Our partial fraction decomposition therefore gives us

$$\begin{aligned}\int \frac{3}{x^3-1}dx&=\int\left(\frac{1}{x-1}-\frac{x+2}{x^2+x+1}\right)dx\\ &=\ln|x-1|+C-\int\frac{x+2}{x^2+x+1}dx\end{aligned}$$

For the right hand side, we'd like to be able to use a u-sub to simplify the fraction, but the derivative of x^2+x+1 is $2x+1$, which we don't have in the numerator. By making some clever rewriting, we can make a multiple of it to do u-sub:

$$\begin{aligned}\int\frac{x+2}{x^2+x+1}dx&=\int\frac{(x+0.5)+1.5}{x^2+x+1}dx\\ &=\int\frac{(1/2)(2x+1)}{x^2+x+1}dx+\int\frac{1.5}{x^2+x+1}dx\\ &=0.5\int\frac{1}{u}du+\int\frac{1.5}{x^2+x+1}dx\\ &=0.5\ln|x^2+x+1|+C+\int\frac{1.5}{x^2+x+1}dx\end{aligned}$$

From here, we can integrate the right side by completing the square and doing trig substitutions after setting v to be the squared term.

$$\begin{aligned}
\int \frac{1.5}{x^2+x+1}dx &= \frac{3}{2}\int \frac{1}{(x+1/2)^2+\frac{3}{4}}dx \\
&= \frac{3}{2}\int \frac{1}{(x+1/2)^2+\frac{3}{4}}dx \\
&= \frac{3}{2}\int \frac{1}{(v)^2+\frac{3}{4}}dv \\
&= \frac{3}{2}\int \frac{\frac{\sqrt{3}}{2}\sec^2(\theta)}{\frac{3}{4}(\tan^2(\theta)+1)}d\theta \\
&= \frac{3}{2}\int \frac{2}{\sqrt{3}}d\theta \\
&= \sqrt{3}\theta \\
&= \sqrt{3}\tan^{-1}\left(\frac{2\sqrt{3}}{3}v\right) \\
&= \sqrt{3}\tan^{-1}\left(\frac{2\sqrt{3}}{3}\left(x+\frac{1}{2}\right)\right)
\end{aligned}$$

Therefore, our final answer is

$$\boxed{\ln|x-1| - 0.5\ln|x^2+x+1| - \sqrt{3}\tan^{-1}\left(\frac{2\sqrt{3}}{3}\left(x+\frac{1}{2}\right)\right)}$$

3. Integration Techniques!

If you're looking for easier problems, do problems in Section 7.5 or in the Review section at the back of the chapter!

$$\int \frac{(x+3)^2}{x}dx$$

In the absence of a good substitution, we can simply expand the numerator and evaluate.

$$\begin{aligned}
\int \frac{(x+3)^2}{x}dx &= \int \frac{x^2+6x+9}{x}dx \\
&= \int \left(x+6+\frac{9}{x}\right)dx \\
&= \boxed{\frac{1}{2}x^2+6x+9\ln x+C}.
\end{aligned}$$

$$\int \sin^2(2x)\sin(x)\cos^4(x)dx$$

We can try to rewrite everything in terms of sines and cosines, as we know how to deal with that. From there, we can use techniques we used in the past to evaluate.

$$\begin{aligned}
\int \sin^2(2x)\sin(x)\cos^4(x)dx &= \int (2\sin(x)\cos(x))^2\sin(x)\cos^4(x)dx \\
&= 4\int sin^2(x)\cos^2(x)\sin(x)\cos^4(x)dx \\
&= 4\int \sin^3(x)\cos^6(x)dx \\
&= 4\int (1-\cos^2(x))\cos^6(x)\sin(x)dx \\
&= 4\int (1-u^2)(u^6)du \\
&= 4\int u^6 - u^8 du \\
&= 4\left(\frac{1}{7}u^7 - 0\frac{1}{9}u^9\right) \\
&= \boxed{4\left(\frac{1}{7}\cos^7(x) - \frac{1}{9}\cos^9(x)\right)}
\end{aligned}$$

$$\int \frac{x\sin(x)}{\cos^2(x)}dx$$

(Put LIATE to good use.)

The hint tells us that we should probably be using IBP, but it's not really clear where we should start. However, LIATE tells us that we should be using algebraic expressions for u and trigonometric functions for dv, so we reluctantly accept $u = x$, $dv = \frac{\sin(x)}{\cos^2(x)}dx$. However, we can observe that the right hand side is actually pretty easy to integrate with u-sub for $u = \cos x$ or by noting that $\frac{\sin(x)}{\cos^2(x)} = \tan(x)\sec(x)$ which is the derivative of $\sec(x)$, so we can do IBP with

$$\begin{aligned}
u &= x \\
du &= dx \\
dv &= \tan(x)\sec(x) \\
v &= \sec(x).
\end{aligned}$$

Plugging into IBP gives us

$$\begin{aligned}
\int \frac{x\sin(x)}{\cos^2(x)}dx &= x\sec(x) - \int \sec(x)dx \\
&= \boxed{x\sec(x) - \ln|\sec(x)+\tan(x)| + C}.
\end{aligned}$$

[The next two are from the MIT Integration Bee, 2007]

$$\int (2\ln x + (\ln x)^2)dx$$

We can definitely solve this problem by integrating both terms in the integral; the left side is easy, and the right side can be done with IBP by setting $dv = 1$. However, there's actually a much better way: the chain rule tells us that $(fg)' = f'g + g'f$, which we can integrate to say

$$fg + C = \int f\ dg + g\ df.$$

Now we note that the left term in the integral looks similar to the derivative of the right side, except we're off by a term x. With this in mind, if we say $f(x) = x$ and $g(x) = (\ln x)^2$, we know that

$$f\ dg = x(2\frac{1}{x}\ln x) = 2\ln x$$
$$g\ df = (\ln x)^2$$

which are the two terms in our integral, so our final integration gives us

$$\int (2\ln x + (\ln x)^2)dx = fg + C = \boxed{x(\ln x)^2 + C}.$$

$$\int \frac{x^{-1/2}}{1+x^{1/3}}dx$$

(Hint: Section 7.4, Problem 45, from your homework.)

As in our homework, we didn't like working with these weird roots and would prefer to express them in terms of integer powers. We fixed this by making the substitution $u^6 = x$, and it works fine here too! Noting that $6u^5du = dx$, we can simply substitute and evaluate:

$$\begin{aligned}\int \frac{x^{-1/2}}{1+x^{1/3}}dx &= \int \frac{u^-3}{1+u^2}6u^5du \\ &= 6\int \frac{u^2}{1+u^2}du \\ &= 6\int \frac{u^2+1-1}{1+u^2} \\ &= 6\int \left(1 - \frac{1}{1+u^2}\right) du \\ &= 6(u - \tan^{-1}(u)) + C \\ &= \boxed{6x^{1/6} - 6\tan^{-1}(x^{1/6}) + C}\end{aligned}$$