MATH 1B DISCUSSION WORKSHEET - 9/6/18

RIEMANN SUMS AND ERRORS

1. Things to know for your quiz! (In terms of approximations anyway)

2. Finding Error Bounds (IMPORTANT)

Find the error bounds for the following functions and intervals for Midpoint, Trapezoidal, and Simpson's.

(1) $\int_{-2}^{5} \left(-2x^3 + 7x^2 + 5x - 12\right)\, dx$. Partitions of length 1. [What's the error for Simpson's? What does that tell us about Cubics?]

Midpoint/Trapezoid:

We should start by looking for the K value. The second derivative of the function is $f''(x) = -12x + 14$, so we're trying to find the maximum magnitude of the function along the interval $[-2, 5]$. To do so, we can simply check the endpoints, because we know that for a linear function the maximum will be at one of the endpoints. $f''(-2) = 38$ and $f''(5) = -46$, so the highest magnitude is 46 which means $K = 46$.

We also know that the partitions are of length 1 and the total length is $5 - (-2) = 7$, so there are $n = \frac{7}{1} = 7$ partitions.

From here, we can simply use our formulas for Midpoint and Trapezoid Rule to find the bounds:

$$|E_{\text{Midpoint}}| \leq \frac{K(b-a)^3}{24n^2} = \frac{46(5-(-2))^3}{24 \cdot 7^2} = \frac{46 \cdot 7^3}{24 \cdot 7^2} = \boxed{\frac{161}{12}}.$$

We also know that the Midpoint Rule's error bound is exactly half that of the Trapezoid Rule's error bound, so we have

$$|E_{\text{Trapezoid}}| \leq \boxed{\frac{161}{6}}.$$

Simpson's:

Note that $f^{(4)} = 0$. This means that we can choose $K = 0$, which tells us that

$$|E_{\text{Simpson's}}| = \boxed{0}.$$

In fact, for any function whose fourth integral is 0 (cubics, quadratics, linear) Simpson's Rule will provide an exact estimate!

(2) $\int_0^4 \left(\frac{-1}{24}(x-2)^4 + x^3\right)\, dx$. Partitions of length 1.

Midpoint/Trapezoid:

We begin by looking for the K value. We note that the second derivative of $f(x) = \frac{-1}{24}(x-2)^4 + x^3$ is $f''(x) = \frac{-1}{2}(x-2)^2 + 6x$. Because we don't know where the vertex of this is, we can take the next derivative to look for critical values. Taking the next derivative give us $f'''(x) = -(x-2) + 6 = 0$, so we can see that the critical value is at 8, which is not in the interval. Therefore, we can see that the maximum second derivative is one of the two ends. Plugging in gives us $f''(0) = -2$ and $f''(4) = 22$, so we choose $K = 22$.

From here, we can simply use our formulas for Midpoint and Trapezoid Rule to find the bounds:

$$|E_{\text{Midpoint}}| \leq \frac{K(b-a)^3}{24n^2} = \frac{22(4-0)^3}{24 \cdot 4^2} = \frac{22 \cdot 4}{24} = \boxed{\frac{11}{3}}.$$

We also know that the Midpoint Rule's error bound is exactly half that of the Trapezoid Rule's error bound, so we have

$$|E_{\text{Trapezoid}}| \leq \boxed{\frac{22}{3}}.$$

Simpson's: We note that $f^{(4)}(x) = -(x-2)$, and because $|f^{(4)}| = |f^{(0)}| = 2$, we have $K = 2$. This tells us that

$$|E_{\text{Simpson's}}| \leq \frac{K(b-a)^5}{180n^4} = \frac{2(4)^5}{180(4)^4} = \frac{8}{180} = \boxed{\frac{2}{45}}.$$

(3) $\int_{-0.5}^{0.5} \sin(\theta^2)\, d\theta$. Partitions of length 0.1. (Hint: these angles aren't that big...)

Midpoint/Trapezoid: We note that $f''(x) = -4\theta^2 \sin(\theta^2) + 2\cos(\theta^2)$. Because the interval contains only numbers very close to 0, we can use the small-angle approximation to state that $\cos(\theta) \approx 0$ and $\sin(\theta) \approx \theta$. This tells us that $f''(x) = -4\theta^2 \sin(\theta^2) + 2\cos(\theta^2) \approx -4\theta^2 \cdot \theta^2 + 0 = -4\theta^4$. On the interval, the maximum for this value is $|-4(0.5)^4| = \frac{1}{4} = K$. Thus, we know that

$$|E_{\text{Midpoint}}| \leq \frac{K(b-a)^3}{24n^2} = \frac{(1/4)(1)^3}{24 \cdot 10^2} = \boxed{\frac{1}{9600}},$$

$$|E_{\text{Trapezoid}}| \leq \boxed{\frac{1}{4800}},$$

Simpson's Rule:

Taking four derivatives, we find $f^{(4)}(x) = 16\theta^4 \sin(\theta^2) - 12\sin(\theta^2) - 48\theta^2\cos(\theta^2)$. Using the small-angle approximation one more time, we see that

$$16\theta^4 \sin(\theta^2) - 12\sin(\theta^2) - 48\theta^2\cos(\theta^2) \approx 16\theta^6 - 12\theta^2,$$

which has a minimum at $f^{(4)}(0.5) = -2.75$. Thus, we have $K = \frac{11}{4}$, which means

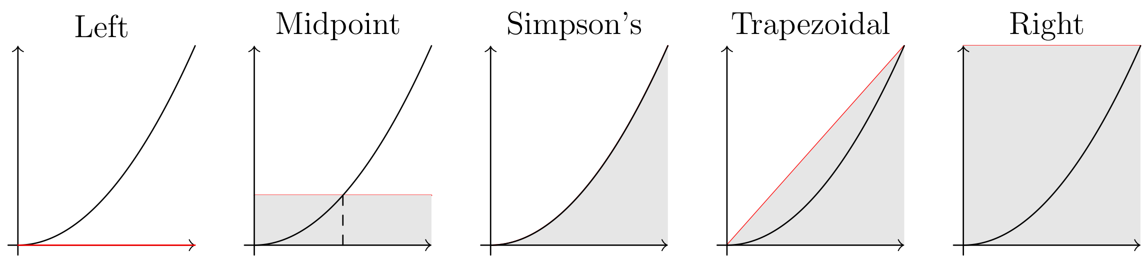
$$|E_{\text{Simpson's}}| \leq \frac{K(b-a)^5}{180n^4} = \frac{(11/4)(1)^5}{180(10)^4} = \boxed{\frac{11}{7200000}}.$$

3. Building Error Intuition

For each of the following functions, rank the five types of approximations in terms from lowest to highest, or state that it can't be properly ranked without knowing more about the partitions. Assume are partitions are regular partitions unless stated otherwise.

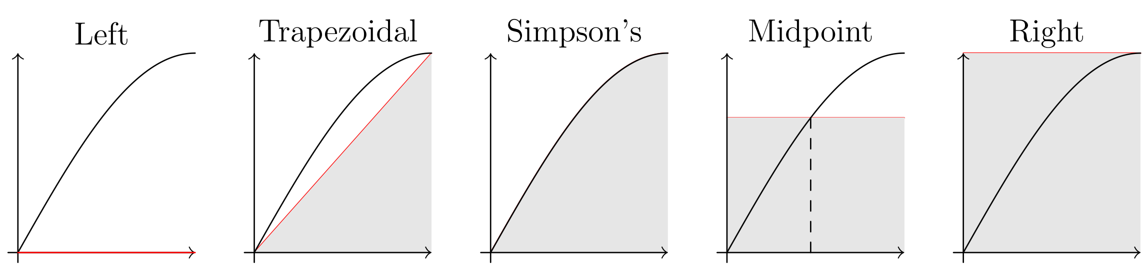
(1) $f(x) = x^2$ for x in $[0, 10000]$.

Because the function continues to grow with the same behavior, we can rank them by simply doing the approximations with a single partition. I have ranked them below, with the approximation that each one gives.



(2) $f(x) = \sin(x)$ for x from 0 to $\pi/2$.

Because the function continues to grow with the same behavior, we can rank them by simply doing the approximations with a single partition. I have ranked them below, with the approximation that each one gives.



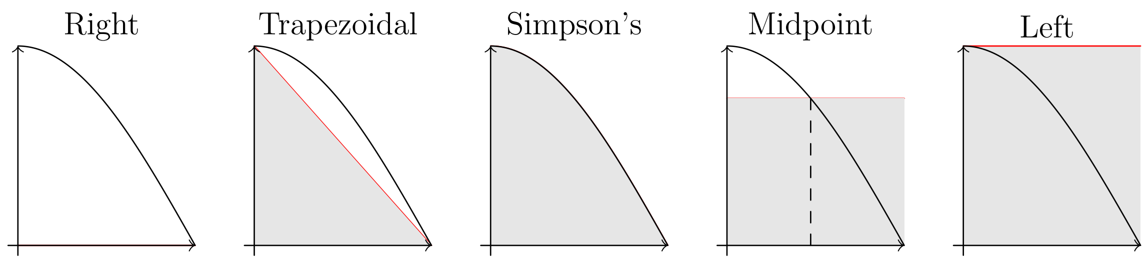
Two notes:

(a) Simpson's isn't actually good enough to perfectly approximate the sine curve, because sine is not a polynomial. I only denoted it as such because Simpson's is always going to be a much better estimate than any of the other methods we're using, at least for regular curves.

(b) It might be difficult to tell that Midpoint is actually better than Trapezoidal just by comparing the two. However, what we do know for sure is that Trapezoidal is an underestimate, and it's not difficult to see that Midpoint is an overestimate. Therefore, Midpoint will be above trapezoidal.

(3) $f(x) = \sin(x)$ for x from $\pi/2$ to π.

Because the function continues to decrease with the same behavior, we can rank them by simply doing the approximations with a single partition. I have ranked them below, with the approximation that each one gives.



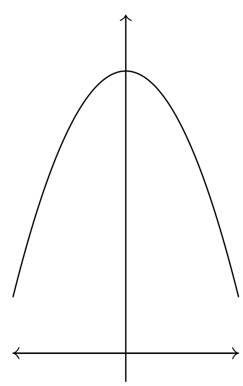
(4) $f(x) = \sin(x)$ for x from 0 to 2π.

As it turns out, every single one of these approximations gives the exact same result for regular partitions: no error! Because the function is entirely symmetric, any estimates made in the interval $[0, \pi]$ has the opposite values as the estimates made in $[\pi, 2\pi]$, so every estimate ends up having value 0 and thus error 0.

4. Unfortunate Riemann Sums

(1) Is it possible to find a partition where both left endpoint approximation and right endpoint approximation give underestimates, but Midpoint Rule gives an overestimate?

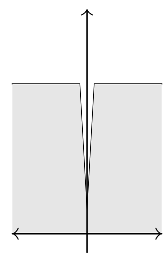
Yes. The following partition does so.



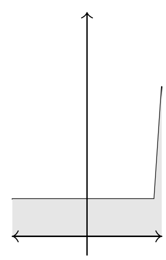
(2) Is it possible to find a partition where both left endpoint approximation and right endpoint approximation give underestimates, but Trapezoidal Rule gives an overestimate?

No. The Trapezoidal rule is an average of the Left Endpoint approximation and the Right Endpoint approximation. Therefore, if both of those approximations give underestimates, then the Trapezoidal rule must as well.

(3) Draw a partition in which left endpoint approximation, right endpoint approximation, and Trapezoidal Rule would provide significantly less error than Midpoint Rule.



(4) Draw a partition in which left endpoint approximation and Midpoint Rule would both provide very little error, but Trapezoidal Rule would give a significant amount of error.



5. Error Bound Exploration (Pretty Challenging)

We never actually went over what the error bounds would look like for left endpoint and right endpoint approximations. It probably isn't surprising to you that they're the same. Can you figure out what they are?

(Hint: the form is pretty similar to our previous error bounds, Furthermore, we're choosing K such that $|f'(x)| \leq K$.)

we note that for any partition, the difference between our approximation and the actual area is maximized when the derivative is maximized the entire length of the partition, because then the graph has deviated farthest from our rectangle on the endpoint. Therefore, the maximum difference is the triangle made with slope K from the left side of the partition to the right. The width of our partition is $\frac{b-a}{n}$, so the height is $K\frac{b-a}{n}$, so our triangle of error has area

$$\frac{1}{2}\left(\frac{b-a}{n}\right)\left(K\frac{b-a}{n}\right) = \frac{K(b-a)^2}{2n^2}.$$

There are n partitions, each of which can have up to $\frac{K(b-a)^2}{2n^2}$ error, so the total possible error is

$$\frac{K(b-a)^2}{2n^2} \cdot n = \boxed{\frac{K(b-a)^2}{2n}}$$