MATH 1B DISCUSSION WORKSHEET - 9/13/18

ARC LENGTH, SURFACE AREA, AND MASSES ANSWERS

1. Comparisons and p-values

Use the Comparison Theorem to determine whether each integral is convergent or divergent.

(1) $\int_1^\infty \sqrt{1+\frac{1}{x^3}}\,dx$

We know that $\frac{1}{x^3} > 0$ for all $x \geq 1$, so we can say that

$$\int_1^\infty \sqrt{1+\frac{1}{x^3}}dx > \int_1^\infty \sqrt{1}dx = \int_1^\infty 1dx,$$

so the integral is divergent.

(2) $\int_0^\infty \frac{dx}{x+e^x}$

Our first instinct is to compare it to x^{-1}, but

$$\int_0^\infty \frac{dx}{x+e^x} < \int_0^\infty \frac{dx}{x},$$

doesn't tell us much. If we try the other comparison, however, we note that

$$\int_0^\infty \frac{dx}{x+e^x} < \int_0^\infty \frac{dx}{e^x},$$

and

$$\begin{aligned}\int_0^\infty e^{-x}dx &= \lim_{a\to\infty}\int_0^a e^{-x} \\ &= \lim_{a\to\infty} -e^{-x}|_0^a \\ &= \lim_{a\to\infty} -e^{-a} - (-e^0) \\ &= 0 - (-1) \\ &= 1.\end{aligned}$$

Thus, the larger integral converges, so the smaller one does as well.

(3) $\int_0^\infty \frac{3x+2\sin(x)}{x^3+5}\,dx$

First, we'll split the fraction up into two separate components. We know that

$$\int_0^\infty \frac{3x+2\sin(x)}{x^3+5}\,dx = \int_0^\infty \frac{3x}{x^3+5}\,dx + \int_0^\infty \frac{2\sin(x)}{x^3+5}\,dx.$$

For both the left and the right hand side, we can find comparisons to integrals that are larger than it:

$$\frac{3x}{x^3+5} \le 3\frac{x}{x^3} = 3\frac{1}{x^2}$$
$$\frac{2\sin(x)}{x^3+5} \le 2\frac{1}{x^3+5}$$

With this, we can see that

$$\begin{aligned}\int_0^\infty \frac{3x+2\sin(x)}{x^3+5}\,dx &= \int_0^\infty \frac{3x}{x^3+5}\,dx + \int_0^\infty \frac{2\sin(x)}{x^3+5}\,dx \\ &\le \int_0^\infty 3\frac{1}{x^2}\,dx + \int_0^\infty 2\frac{1}{x^3+5}\,dx \\ &= 3\int_0^\infty \frac{1}{x^2}\,dx + 2\int_0^\infty \frac{1}{x^3+5}\,dx \\ &\le 3\int_0^\infty \frac{1}{x^2}\,dx + 2\int_0^\infty \frac{1}{x^3}\,dx\end{aligned}$$

Because $3 > 2 > 1$, by the p-test, we know that both of these integrals converge. Therefore, the sum of two convergent integrals is greater than our integral, so our integral is convergent.

(4) $\int_0^\infty 2^{\left(-x^2\right)}\,dx$

We'd like to compare this to a function that we know converges, like $\int_0^\infty 2^{(-x)}\,dx$. However, the behavior is different on different intervals: $2^{(-x)} > 2^{-x}$ when $x < 1$, but $2^{(-x)} < 2^{-x}$ when $x > 1$. With this in mind, we can rewrite our integral and then compare:

$$\begin{aligned}\int_0^\infty 2^{\left(-x^2\right)}\,dx &= \int_0^1 2^{\left(-x^2\right)}\,dx + \int_1^\infty 2^{\left(-x^2\right)}\,dx \\ &\le \int_0^1 2^{\left(-x^2\right)}\,dx + \int_1^\infty 2^{-x}\,dx\end{aligned}$$

Now, we know that the right side is convergent, so what's left is to demonstrate that the left side is convergent as well. However, there is no number x such that 2^{-x^2} is undefined, which means that the left side is not an improper integral and thus must have a finite value. Therefore, both sides of the sum are convergent, so the original integral is convergent as well.

2. Arc Length

[Exercise from Pauls Online Notes]

(1) Find the length of $x = \left(\frac{3y}{2}\right)^{2/3} + 1$ between $0 \leq y \leq 2\sqrt{3}$.

Note that if $g(y) = \left(\frac{3y}{2}\right)^{2/3} + 1$, then

$$g'(y) = \frac{3}{2} \cdot \frac{2}{3} \left(\frac{3y}{2}\right)^{-1/3} = \left(\frac{3y}{2}\right)^{-1/3},$$

so our integral for arc length becomes

$$\int_0^{2\sqrt{3}} ds = \int_0^{2\sqrt{3}} \sqrt{1 + \left(\left(\frac{3y}{2}\right)^{-1/3}\right)^2}\, dy$$

Evaluating this integral gives

$$\begin{aligned}
\int_0^{2\sqrt{3}} ds &= \int_0^{2\sqrt{3}} \sqrt{1 + \left(\left(\frac{3y}{2}\right)^{-1/3}\right)^2}\, dy \\
&= \int_0^{2\sqrt{3}} \sqrt{1 + \left(\frac{3y}{2}\right)^{-2/3}}\, dy \\
&= \int_0^{2\sqrt{3}} \sqrt{1 + \frac{1}{\left(\frac{3y}{2}\right)^{2/3}}}\, dy \\
&= \int_0^{2\sqrt{3}} \sqrt{\frac{\left(\frac{3y}{2}\right)^{2/3} + 1}{\left(\frac{3y}{2}\right)^{2/3}}}\, dy \\
&= \int_0^{2\sqrt{3}} \frac{\sqrt{\left(\frac{3y}{2}\right)^{2/3} + 1}}{\left(\frac{3y}{2}\right)^{1/3}}\, dy \\
&= \int_1^4 \sqrt{u}\, du \left[u = \left(\frac{3y}{2}\right)^{2/3} + 1, du = \left(\frac{3y}{2}\right)^{-1/3} dy\right] \\
&= \frac{2}{3} u^{3/2}|_1^4 \\
&= \frac{2}{3}(4)^{3/2} - \frac{2}{3}(1)^{3/2} \\
&= \frac{16}{3} - \frac{2}{3} \\
&= \boxed{\frac{14}{3}}
\end{aligned}$$

(2) Redo the previous problem in the form $y = f(x)$ instead. (Rewrite the previous equation to in terms of $y = f(x)$, look at what range the value of x changes by in that span of y-values, and find the arc length again.) Do we expect the answers to be the same?

Knowing that the original equation was $x = \left(\frac{3y}{2}\right)^{2/3} + 1$, we can solve to find

$$\begin{aligned} x &= \left(\frac{3y}{2}\right)^{2/3} + 1 \\ x - 1 &= \left(\frac{3y}{2}\right)^{2/3} \\ (x-1)^{3/2} &= \frac{3y}{2} \\ \frac{2}{3}(x-1)^{3/2} &= y \end{aligned}$$

Now, we must find out the range of x that the arc travels through. We are looking at $0 \le y \le 2\sqrt{3}$ for $x = \left(\frac{3y}{2}\right)^{2/3} + 1$, so the lower bound is

$$\left(\frac{3(0)}{2}\right)^{2/3} + 1 = 1$$

and the upper bound is

$$\left(\frac{3(2\sqrt{3})}{2}\right)^{2/3} + 1 = 4.$$

Thus, the arc length can be found by integrating normally:

$$\begin{aligned} \int_1^4 ds &= \int_1^4 \sqrt{1 + \left[\frac{d}{dx}\left(\frac{2}{3}(x-1)^{3/2}\right)\right]^2}\, dx \\ &= \int_1^4 \sqrt{1 + \left[(x-1)^{1/2}\right]^2}\, dx \\ &= \int_1^4 \sqrt{1 + (x-1)}\, dx \\ &= \int_1^4 \sqrt{x}\, dx \\ &= \frac{2}{3}x^{3/2}|_1^4 \\ &= \frac{2}{3}(4)^{3/2} - \frac{2}{3}(1)^{3/2} \\ &= \frac{16}{3} - \frac{2}{3} \\ &= \boxed{\frac{14}{3}} \end{aligned}$$

We can see that the arc lengths end up evaluating to the same number, as expected because it is the same arc.

3. Surface Area

(1) Find the surface area of the rotation of the curve $y = 4-x$ for $1 \leq x \leq 3$ about the x-axis.

We are rotating about the x-axis, which means we will be using the following formula for surface area:

$$\int_a^b 2\pi|f(x) - y_0|\sqrt{1 + [f'(x)]^2}\, dx.$$

Plugging in, we get

$$\begin{aligned}\int_a^b 2\pi|f(x) - y_0|\sqrt{1 + [f'(x)]^2}\, dx &= \int_1^3 2\pi|4 - x|\sqrt{1 + [-1]^2}\, dx \\ &= 2\pi \int_1^3 (4 - x)\sqrt{2}\, dx \\ &= 2\sqrt{2}\pi \left(4x - \frac{1}{2}x^2\right)\Big|_1^3 \\ &= 2\sqrt{2}\pi\left((12 - 4.5) - (4 - 0.5)\right) \\ &= 2\sqrt{2}\pi\,(4) \\ &= \boxed{8\pi\sqrt{2}}\end{aligned}$$

(2) Find the surface area of the rotation of the curve $y = 4 - x$ for $1 \leq y \leq 3$ about the line $y = -2$.

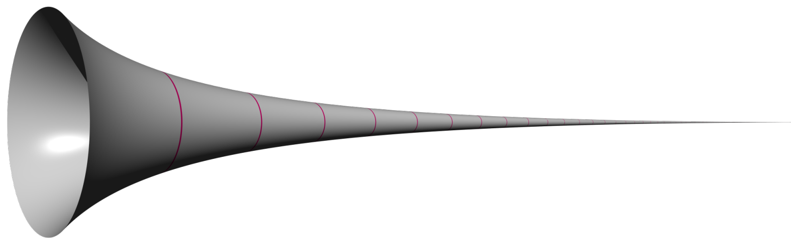
We are rotating around a line parallel to the x-axis, so the formula we use will be the same:

$$\int_a^b 2\pi|f(x) - y_0|\sqrt{1 + [f'(x)]^2}\, dx.$$

However, this time, we must adjust our "radius" term to match the radius of rotation. Because we are rotating around $y = -2$, our radius is actually 2 larger than the function's height, which means our integral will actually be

$$\begin{aligned}\int_a^b 2\pi|f(x) + 2|\sqrt{1 + [f'(x)]^2}\, dx &= \int_1^3 2\pi|4 - x + 2|\sqrt{1 + [-1]^2}\, dx \\ &= 2\pi \int_1^3 (6 - x)\sqrt{2}\, dx \\ &= 2\sqrt{2}\pi \left(6x - \frac{1}{2}x^2\right)\Big|_1^3 \\ &= 2\sqrt{2}\pi\left((18 - 4.5) - (6 - 0.5)\right) \\ &= 2\sqrt{2}\pi\,(8) \\ &= \boxed{16\pi\sqrt{2}}\end{aligned}$$

(3) Now that you know how to find surface area, we can look at Gabriel's Horn one more time. Gabriel's Horn is defined as the rotation of the curve $y = \frac{1}{x}$ for $1 \leq x \leq \infty$ about the x-axis.



(a) Find the length of the curve. (This one is pretty obvious.)

It's pretty clear that the curve goes on forever, so the length is infinite.

(b) Find the volume.

We know that the volume can be found with $\int_a^b \pi r^2 dx$. We know that $r = \frac{1}{x}$, so we can see that

$$\begin{aligned}\int_1^\infty \pi r^2 &= \int_1^\infty \pi \left(\frac{1}{x}\right)^2 \\ &= \lim_{a\to\infty} \int_1^a \pi \left(\frac{1}{x}\right)^2 \\ &= \lim_{a\to\infty} -\pi\frac{1}{x}|_1^a \\ &= \lim_{a\to\infty} -\pi\frac{1}{a} + \pi\frac{1}{1} \\ &= 0 + \pi \\ &= \boxed{\pi}.\end{aligned}$$

(c) Find the surface area.

The surface area is given by

$$\int_1^\infty 2\pi \left|\frac{1}{x} - 0\right| \sqrt{1 + [\ln(x)]^2}\, dx.$$

We can see that $[\ln(x)]^2 \geq 0$ when $x \geq 1$, which tells us that $\sqrt{1 + [\ln(x)]^2} \geq 1$ for $x \geq 1$. This tells us that

$$2\pi \left|\frac{1}{x} - 0\right| \sqrt{1 + [\ln(x)]^2} > 2\pi \left|\frac{1}{x} - 0\right| > \frac{1}{x}$$

for all $x \geq 1$, which tells us that

$$\int_1^\infty 2\pi \left|\frac{1}{x} - 0\right| \sqrt{1 + [\ln(x)]^2}\, dx > \int_1^\infty \frac{1}{x}\, dx.$$

The second integral diverges, so the first one must diverge as well. The surface area is thus $\boxed{\text{infinite}}$.

4. Centers of Mass

Finding the Center of Mass of a region bounded by $f(x)$ on top and $g(x)$ below consists of these steps:

- First, find the moments M_x and M_y. These denote the tendency of the region to rotate around the x or y axis, respectively. Their equations are:
$$M_x = \frac{1}{2}\int_a^b y \cdot f(x) - y \cdot g(x)\,dx = \frac{1}{2}\int_a^b (f(x))^2 - (g(x))^2\,dx$$
$$M_y = \int_a^b xf(x) - xg(x)\,dx$$
- Solve for the total mass M. This is simply the area between the two curves.
- The center of mass, $(\overline{x}, \overline{y})$ can be found with:
$$(\overline{x}, \overline{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right).$$

(1) Find the center of mass of the semicircle made by $y = \sqrt{4 - x^2}$ and the x-axis.

First, we know that M is the area of the semicircle. The semicircle has radius 2, so we know that the area is 4π.

Now, we must solve for the moments across the x and y axis. The moment M_x can be found with the integral
$$\begin{aligned}\frac{1}{2}\int_a^b (f(x))^2 - (g(x))^2\,dx &= \frac{1}{2}\int_{-2}^{2} \left(\sqrt{4 - x^2}\right)^2 - 0\,dx \\ &= \frac{1}{2}\int_{-2}^{2} 4 - x^2\,dx \\ &= \frac{1}{2}\left(4x - \frac{1}{3}x^3\right)\Big|_{-2}^{2} \\ &= \frac{1}{2}((8 - (8/3)) - (-8 - (-8/3))) \\ &= \frac{1}{2}\left(\frac{16}{3} + \frac{16}{3}\right) \\ &= \frac{16}{3}.\end{aligned}$$

From this, we can tell that
$$\overline{y} = \frac{M_x}{M} = \frac{16/3}{4\pi} = \boxed{\frac{4}{3\pi}}$$

Now, we could solve for M_y, but we also know that the shape is exactly symmetrical across the y-axis, so we know that the center of mass's x-coordinate will be $\overline{x} = 0$.

Thus, we find
$$\boxed{(\overline{x}, \overline{y}) = \left(0, \frac{4}{3\pi}\right)}$$

(2) Determine the center of mass of the region bounded by the parabola $y = x^2$ and the line $y = 9$.

We know that the parabola intersects the line at $x = \pm 3$, so the area is

$$\begin{aligned} M &= \int_a^b f(x) - g(x) \\ &= \int_{-3}^{3} 9 - x^2 \\ &= \left(9x - \frac{1}{3}x^3\right)\Big|_{-3}^{3} \\ &= (27-9) - (-27+9) \\ &= 36 \end{aligned}$$

Now, we can proceed. It's clear that $M_y = 0$ because the graph is once again symmetrical across the y-axis, and we know that the x-coordinate of the center of mass should be 0.

We know that the formula for M_x is given by

$$M_x = \frac{1}{2}\int_a^b (f(x))^2 - (g(x))^2\, dx,$$

so we can simply plug in and evaluate:

$$\begin{aligned} \frac{1}{2}\int_a^b (f(x))^2 - (g(x))^2\, dx &= \frac{1}{2}\int_{-3}^{3} (9)^2 - (x^2)^2\, dx \\ &= \frac{1}{2}\int_{-3}^{3} 81 - x^4\, dx \\ &= \frac{1}{2}\left(81x - \frac{1}{5}x^5\right)\Big|_{-3}^{3} \\ &= \frac{1}{2}\left(\left(81(3) - \frac{1}{5}3^5\right) - \left(81(-3) - \frac{1}{5}(-3)^5\right)\right) \\ &= \frac{1}{2}\left(2(243) - \frac{2}{5}3^5\right) \\ &= (243) - \frac{1}{5}(243) \\ &= 194.4. \end{aligned}$$

We now know that $M_x = 194.4$, so $\overline{y} = \frac{194.4}{36} = 5.4$, so we end up with

$$\boxed{(\overline{x}, \overline{y}) = (0, 5.4)}$$