MATH 1B DISCUSSION WORKSHEET - 9/20/18

MIDTERM REVIEW ANSWERS!

[Note that I haven't actually seen the midterm so this is really just me guessing.]

1. Section 7.1

(1) Evaluate

$$\int_0^{\pi/2} e^x \cos(x)\, dx.$$

We will use integration by parts. Following LIATE, we see that we should choose $u = \cos(x)$ and $dv = e^x$. This gives us the following:

$$u = \cos(x), du = -\sin(x)\, dx$$
$$v = e^x, dv = e^x\, dx$$

Doing integration by parts gives us:

$$\begin{aligned}\int_0^{\pi/2} e^x \cos(x)\, dx &= e^x \cos(x)\big|_0^{\pi/2} - \int_0^{\pi/2} -\sin(x)e^x\, dx \\ &= e^x \cos(x)\big|_0^{\pi/2} + \int_0^{\pi/2} \sin(x)e^x\, dx\end{aligned}$$

We'll do integration by parts again on the integral on the right hand side of the equation. Again by LIATE, we will assign

$$u = \sin(x), du = \cos(x)\, dx$$
$$v = e^x, dv = e^x\, dx$$

Plugging in to our original equation, we find:

$$\begin{aligned}\int_0^{\pi/2} e^x \cos(x)\, dx &= e^x \cos(x)\big|_0^{\pi/2} + \left[e^x \sin(x)\big|_0^{\pi/2} - \int_0^{\pi/2} e^x \cos(x)\, dx\right] \\ 2\int_0^{\pi/2} e^x \cos(x)\, dx &= e^x \cos(x)\big|_0^{\pi/2} + e^x \sin(x)\big|_0^{\pi/2} \\ 2\int_0^{\pi/2} e^x \cos(x)\, dx &= e^{\pi/2}\cos(\pi/2) - e^0\cos(0) + e^{\pi/2}\sin(\pi/2) - e^0\sin(0) \\ 2\int_0^{\pi/2} e^x \cos(x)\, dx &= 0 - 1 + e^{\pi/2} - 0 \\ \int_0^{\pi/2} e^x \cos(x)\, dx &= \boxed{\frac{1}{2}\left(e^{\pi/2} - 1\right)}\end{aligned}$$

2. Section 7.2 - 7.3

If you need more practice with 7.2 and 7.3, I'd recommend working on my worksheet from August 28! The problems there are essentially everything you need to be able to do from those two sections. That worksheet, and its solutions, are available on bCourses.

3. Section 7.4

(2) Evaluate
$$\int \frac{x^4+x^3+x^2+1}{x^2+x-2}\,dx.$$

We can see that the numerator has higher degree than the denominator, so we should start by doing polynomial division to divide. Doing so, we find that
$$\int \frac{x^4+x^3+x^2+1}{x^2+x-2}\,dx = \int x^2+3+\frac{-3x+7}{x^2+x-2}\,dx.$$

From here, we recognize that the right side should be integrated through Partial Fraction Decomposition. To do so, we begin by rewriting the fraction as the sum of two smaller fractions:
$$\frac{-3x+7}{x^2+x-2} = \frac{-3x+7}{(x-1)(x+2)} = \frac{A}{x-1}+\frac{B}{x+2}.$$

From here, we solve for A and B through the equation:
$$\frac{-3x+7}{x^2+x-2} = \frac{A}{x-1}+\frac{B}{x+2}$$
$$-3x+7 = A(x+2)+B(x-1)$$

Now, we can use this to set up a system of equations, but we can also choose smart values of x to help save us some steps. For example, if we choose $x=1$, we find
$$-3(1)+7 = A(1+2)+B(1-1), \text{ so } 3A=4 \implies A=\frac{4}{3}.$$

Likewise, plugging in $x=-2$ gives
$$-3(-2)+7 = A(-2+2)+B(-2-1), \text{ so } 13=-3B \implies B=\frac{-13}{3}.$$

We can then see that our integral can be rewritten as:
$$\int \frac{x^4+x^3+x^2+1}{x^2+x-2}\,dx = \int x^2+3+\frac{4/3}{x-1}+\frac{-13/3}{x+2}$$
$$= \boxed{\frac{1}{3}x^3+3x+\frac{4}{3}\ln|x-1|-\frac{13}{3}\ln|x+2|+C}$$

4. Section 7.7

(3) Give an error bound for the Midpoint Sum approximation of the function

$$f(x) = 3x^2 + x^2\sin(x) - 4x\sin(x) - 2\sin(x) + 4x\cos(x) - 8\cos(x)$$

in 3 partitions in the interval $[0, 6]$.

We should begin by finding the K value. The second derivative of the function can be found to be

$$f''(x) = 6 - (x-2)^2\sin(x).$$

Now, we know that $\sin(x)$ falls in the interval $[-1, 1]$, but we can see that the magnitude is higher when $\sin(x) = -1$, so we can rewrite the second derivative as

$$f''(x) = 6 - (x-2)^2\sin(x) \le 6 - (x-2)^2(-1) = 6 + (x-2)^2.$$

We note that the function has a minimum at $x = 2$, so its maxima will be found at the ends of the interval. Plugging in the two ends of the interval, we see that it would be maximized at $x = 6$, and we find

$$f''(6) = 6 + (4)^2 = 22.$$

Our formula is

$$|E_M| \le \frac{K(b-a)^3}{24n^2} = \frac{22(6-0)^3}{24(3)^2} = \boxed{22}$$

5. Section 7.8

(4) Using the comparison test, determine whether

$$\int_1^\infty \frac{xe^{-2x}}{x^3+1}\,dx$$

converges or diverges.

Our first step is to notice that $x > e^{-2x}$, which means that the behavior of our integrand is probably pretty similar to $\frac{x}{x^3} = \frac{1}{x^2}$, so we think that our integral probably converges. Knowing that, we know that we must find an integral larger than the one given that still converges. We do so as follows:

$$\begin{aligned}\int_1^\infty \frac{xe^{-2x}}{x^3+1}\,dx &= \int_1^\infty \frac{x}{(x^3+1)e^{2x}}\,dx \\ &\le \int_1^\infty \frac{x}{x^3+1}\,dx \\ &\le \int_1^\infty \frac{x}{x^3}\,dx \\ &= \int_1^\infty \frac{1}{x^2}\,dx.\end{aligned}$$

Because $p = 2$, we note that this integral converges, so we are done.

(5) Evaluate

$$\int_0^\infty xe^{-x}\,dx.$$

[For bonus points, find an integral to compare it to once you figure out whether it converges.]

This integral isn't particularly difficult to evaluate, but we must be careful to follow proper procedure during evaluation. The proper procedure is as follows:

$$\begin{aligned}
\int_0^\infty xe^{-x}\,dx &= \lim_{a\to\infty}\int_0^a xe^{-x}\,dx \\
&= \lim_{a\to\infty}\int_0^a udv[u=x, dv=e^{-x}dx] \\
&= \lim_{a\to\infty} -xe^{-x}|_0^a - \int_0^a -e^{-x}dx \\
&= \lim_{a\to\infty} -ae^{-a} - (-0e^{-0}) - e^{-x}|_0^a \\
&= \lim_{a\to\infty} -ae^{-a} - (-0e^{-0}) - (e^{-a} - e^{-0}) \\
&= \lim_{a\to\infty} -ae^{-a} - e^{-a} + 1 \\
&= \lim_{a\to\infty} -\frac{a}{e^a} - 0 + 1 \\
&\overset{H}{=} \lim_{a\to\infty} -\frac{1}{e^a} + 1 \\
&= \boxed{1}.
\end{aligned}$$

Note that L'Hopital's Rule was invoked, and that the limit notation was preserved throughout the entire problem. This is absolutely necessary to get credit for improper integral problems.

[As for the bonus question, note that $xe^{-x} < x^{-2}$ for almost all x. Can you prove that?]

6. Section 8.2

(6) Consider the region bounded by $f(x) = \sqrt{1+e^x}$ and the x-axis between $0 \le x \le 1$. Find the area of the surface obtained by rotating the curve f(x) about the x-axis.

In this case, we are rotating about the line $y = 0$, so we will use the equation

$$SA = \int_a^b 2\pi |y - (0)|\, ds.$$

Plugging in $ds = \sqrt{1 + [f'(x)]^2}\, dx$, we can simply evaluate:

$$\begin{aligned}
SA &= \int_a^b 2\pi |y|\, ds \\
&= 2\pi \int_0^1 \sqrt{1+e^x}\sqrt{1+[f'(x)]^2}\, dx \\
&= 2\pi \int_0^1 \sqrt{1+e^x}\sqrt{1+\left[\frac{e^x}{2\sqrt{1+e^x}}\right]^2}\, dx \\
&= 2\pi \int_0^1 \sqrt{1+e^x}\sqrt{1+\frac{e^{2x}}{4(1+e^x)}}\, dx \\
&= 2\pi \int_0^1 \sqrt{1+e^x}\sqrt{\frac{4(1+e^x)}{4(1+e^x)}+\frac{e^{2x}}{4(1+e^x)}}\, dx \\
&= 2\pi \int_0^1 \sqrt{1+e^x}\sqrt{\frac{4+4e^x+e^{2x}}{4(1+e^x)}}\, dx \\
&= 2\pi \int_0^1 \sqrt{(1+e^x)\frac{(e^x+2)^2}{4(1+e^x)}}\, dx \\
&= 2\pi \int_0^1 \sqrt{\frac{(e^x+2)^2}{4}}\, dx \\
&= 2\pi \int_0^1 \frac{1}{2}(e^x+2)\, dx \\
&= 2\pi \frac{1}{2}\left[e^x + 2x\right]\big|_0^1 \\
&= 2\pi \frac{1}{2}[e + 2 - (1)] \\
&= \boxed{\pi(e+1)}
\end{aligned}$$

7. Section 8.3

For hydrostatic forces, Section 8.3 Questions 3-11 should get you all the practice you need!

Moments and Centers of Mass are literally just plug and chug. Memorize the equations(!!), and remember that wherever there's an x term or $f(x)$ term it can be replaced with y and $g(y)$, respectively, but beyond that these problems are super straightforward. For practice, look at my worksheet from September 13 or Questions 25-35 in the textbook.

8. Section 8.4

I don't want to give questions on the biology section because honestly I have no idea what he'd even ask on the exam about them, but if you know the formulas from the sheet and what they mean I'm sure you'll be fine. Do the homework as well!

(7) [Question 5 from Section 8.4] A demand curve is given by $p(x) = 450/(x+8)$. Find the consumer surplus if the selling price is \$10.

This question requires you to be familiar with the equation:

$$CS = \int_a^b p(x) - P.$$

We start with the obvious: because P stands for price, we already have P=10. Furthermore, it's clear that $p(x) = 450/(x+8)$. However, we do still have to find the bounds! Note that the consumer surplus is the region between the price and and the demand curve, so the value b in the bound is where the $p(x) = P$.

$$\begin{aligned} p(x) &= P \\ \frac{450}{x+8} &= 10 \\ 45 &= x+8 \\ x &= 37. \end{aligned}$$

Now that we know b, we can set up our integral! We have

$$\begin{aligned} CS &= \int_a^b p(x) - P \\ &= \int_0^{37} \frac{450}{x+8} - 10 \\ &= [450 \ln|x+8| - 10x]|_0^{37} \\ &= 450\ln(45) - 370 - 450\ln(8) \\ &= \boxed{450\ln(45/8) - 370} \\ &\approx \$407.25 \end{aligned}$$

9. Section 8.5

(8) Consider the function

$$f(x) = \begin{cases} Ae^{-cx} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the value of A such that f is a Probability Density Function. [What's the name of this type of PDF?]

We should know from class that $A = c$. [If not, read the class notes from 9/19. You can also solve for this.] This is an **exponential probability distribution**.

(b) Find the value d such that

$$Pr[x > d] = 0.5.$$

In other words, find the value of d, in terms of c, such that there's an exactly 50 percent chance that the x from our random variable is greater than d. [What's the name of this value d?]

We know that this question is equivalent to finding d such that

$$\int_d^\infty ce^{-cx}\, dx = 0.5.$$

Solving, we have:

$$\begin{aligned} 0.5 &= \int_d^\infty 0.5e^{-0.5x}\, dx \\ 0.5 &= -e^{-cx}|_d^\infty \\ 0.5 &= 0 - \left(-e^{-cd}\right) \\ 0.5 &= e^{-cd} \\ \ln|1/2| &= -cd \\ \ln|2| &= cd \\ d &= \boxed{\frac{1}{c}\ln(2)} \end{aligned}$$

This value is the **median**. Can you see why?