MATH 1B DISCUSSION WORKSHEET - 9/27/18

TALKING ABOUT SEQUENCES! ANSWERS TO PAGE 2.

1. Limit Rules

If $\lim_{n\to\infty} a_n = L_a$ and $\lim_{n\to\infty} b_n = L_b$, and k is some constant, then

(1) $\lim_{n\to\infty}(a_n + b_n) = L_a + L_b$

(2) $\lim_{n\to\infty}(a_n - b_n) = L_a - L_b$

(3) $\lim_{n\to\infty}(a_n * b_n) = L_a * L_b$

(4) $\lim_{n\to\infty}(a_n \div b_n) = L_a \div L_b$

(5) $\lim_{n\to\infty} k(a_n) = kL_a$

(6) $\lim_{n\to\infty}(a_n)^k = (L_a)^k$

(7) If $\lim_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} |a_n| = 0$

(8) If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

(1) What must be true for us to properly use equation (4)?

L_b can not equal 0, because otherwise we'd be dividing by 0.

(2) Equations (7) and (8) aren't always true if the limit was some non-zero number. Find counterexamples for them.

The sequence $\{a_n\} = \{2, -2, 2, -2, ...\}$ has no limit, but the sequence $\{|a_n|\} = \{2, 2, 2, 2, 2, 2...\}$ clearly has limit 2.

The sequence $\{-2, -2, -2, -2, ...\}$ has limit -2, but the absolute value of that sequence, $\{2, 2, 2, 2, 2, ...\}$ has limit $2 \neq -2$.

2. A single all-inclusive example

Consider the sequence $\{a_n\}$ defined by $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2a_n}$.

(1) Write out a_2, then find its value by plugging into a calculator.
$a_2 = \sqrt{2\sqrt{2}} = 1.682$.

(2) Write out a_3, then find its value by plugging into a calculator.
$a_3 = \sqrt{2\sqrt{2\sqrt{2}}} = 1.834$.

(3) Show that $a_n > 1$ for all n.

Clearly a_1 isn't less than or equal to 1. What would happen if there was some a_k less than 1? In that case, there must be a first term in the sequence smaller than 1. Let's say that a_k is the first term in the sequence such that $a_k \leq 1$. Then this means that $a_k = \sqrt{2a_{k-1}} \leq 1$, so

$$\begin{aligned} \sqrt{2a_{k-1}} &\leq 1 \\ 2a_{k-1} &\leq 1 \\ a_{k-1} &\leq \frac{1}{2} \end{aligned}$$

But we already said that a_k is the first term in the sequence less than or equal to 1, so how can a_{k-1} be less than or equal to 0.5? This tells us that we can't find a "first term" in the sequence that's less than or equal to 1, so there is no term less than or equal to 1.

(4) Show that $a_n < 2$ for all n.

Let's rewrite our sequence as a function. Clearly $a_1 = 2^{1/2}$, so we can see that

$$\begin{aligned} a_2 &= \sqrt{2 \cdot 2^{1/2}} \\ &= \sqrt{2^{3/2}} \\ &= 2^{3/4} \\ a_3 &= \sqrt{2 \cdot 2^{3/4}} \\ &= \sqrt{2^{7/2}} \\ &= 2^{7/8} \\ a_4 &= \sqrt{2 \cdot 2^{7/8}} \\ &= \sqrt{2^{15/8}} \\ &= 2^{15/16} \\ &\dots \end{aligned}$$

So we can see then that

$$a_n = 2^{\frac{2^n - 1}{2^n}}$$

With this in mind, we can see that $a_n \geq 2$ only if $\frac{2^n-1}{2^n} \geq 1$, which is never true.

(5) The previous two proofs show that $\{a_n\}$ is bounded.

(6) Use the results from (3) and (4) to demonstrate that $a_{n+1} > a_n$ for all n.

We have seen that $1 < a_n < 2$. If we manipulate the inequality $a_{n+1} > a_n$, we find

$$\begin{aligned} a_{n+1} &> a_n \\ \sqrt{2a_n} &> a_n \\ 2a_n &> (a_n)^2 \\ 2 &> a_n \end{aligned}$$

but because we know that $a_n < 2$, this inequality will always be true.

(7) This demonstrates that $\{a_n\}$ is monotonic and increasing.

(8) If statements (5) and (7) are true, then $\{a_n\}$ is convergent.

(9) (Bonus) Show that $\lim_{n\to\infty} a_n = \sqrt{2\sqrt{2\sqrt{2\sqrt{2...}}}} = 2$.
Assume that we have some L such that

$$\sqrt{2\sqrt{2\sqrt{2\sqrt{2...}}}} = L.$$

Then

$$\begin{aligned} \sqrt{2\sqrt{2\sqrt{2\sqrt{2...}}}} &= L \\ \sqrt{2L} &= L \\ 2L &= L^2 \\ 2 &= L, \end{aligned}$$

as desired.

Alternatively, note that our function is given by

$$a_n = 2^{\frac{2^n-1}{2^n}},$$

and

$$\lim_{n\to\infty} 2^{\frac{2^n-1}{2^n}} = \lim_{n\to\infty} 2^{1-\frac{1}{2^n}} = 2^1 = 2.$$