MATH 1B DISCUSSION WORKSHEET - 10/2/18

SERIES AND INDUCTION

1. Induction Problems

(1) Prove that, for all odd numbers a, $a^2 - 1$ is divisible by 8.

(2) Prove that

$$\sum_{i=1}^{n} \frac{1}{n(n+1)} = \frac{n}{n+1}$$

(3) The Fibonacci numbers are the sequence $\{f_n\}$ defined by the recurrent $f_{n+2} = f_{n+1} + f_n$ and $f_1 = 1$ and $f_2 = 1$. Use induction to prove the following:
  (a) Suppose that ϕ is a number satisfying the property $\phi^2 = \phi + 1$ (This ϕ is the golden ratio.) Prove that $f_n \geq \phi^{n-2}$ for all n.

  (b) Prove that $f_1 + f_2 + f_3 + ... + f_n = f_{n+2} - 1$.

  (c) Prove that $f_1 + f_3 + f_5 + ... + f_{2n-1} = f_{2n}$.

(4) The harmonic numbers $\{h_n\}$ is defined by $h_n = \frac{1}{n}$.
  (a) Use induction to prove that

$$\sum_{i=1}^{2^n} h_n \geq 1 + \frac{n}{2}$$

  for all n.

  (b) Use this to prove that the sum of the harmonic numbers diverges.

(5) (Stolen from Morgan Weiler) I recently paid a visit to a magical island, inhabited by 99 highly intelligent ions and a single goat. While there, I decided to teach the lions mathematial induction. They seemed fascinated by the subject, and at first I wasn't sure what use a lion would have for induction. Then, when my tour ship left the island, the captain's first mate told me a story...

He explained that, on that island, the lions eat grass instead of goat, because the goat has magical properties. Indeed, should a lion eat the magical goat, then overnight, that lion would BECOME the magical goat. Until now, this had seemed to pose a paradox to the lions: if they should eat the goat, then what would stop them from being eaten the next day? And if none of the lions should eat the goat, then being the goat would be perfectly safe, so maybe they should eat the goat. But my lecture on induction solved their conundrum!

To be clear: each lion's first priority is to survive, and their second is to eat goat rather than grass. This means that they will only eat the goat if they know they will not be eaten once they become the goat. Also, except when the goat is eaten, the lion population remains perfectly stable. And the lions can not share the meat of one goat.

When I return to the island, how many lions will I find?

2. Series

(1) Write down for which values of positive r each of the following series converge, and a formula for the sum when they do converge:
  (a)
$$\sum_{n=1}^{7} \left(\frac{1}{r}\right)^n$$
  (b)
$$\sum_{n=1}^{\infty} \left(\frac{1}{r}\right)^n$$
  (c)
$$\sum_{n=4}^{\infty} \left(\frac{1}{r}\right)^n$$

(2) Determine whether the following series converge or diverge. If it converges, find its sum.
  (a)
$$\sum_{n=1}^{\infty} \frac{1}{n}$$
  (b)
$$\sum_{n=1}^{\infty} \sqrt{2}^{-k}$$
  (c)
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots$$
  (d)
$$\sum_{n=1}^{\infty} \frac{2^n + 3^n}{5^n}$$