MATH 1B DISCUSSION WORKSHEET - 10/4/18

COMPARISON TEST SOLUTIONS

Determine whether the following series converge or diverge. You may use Comparison Test, Limit Comparison Test, Divergence Test, p-Test, or the fact that geometric series always converge when the ratio is less than 1.

(1) $\sum_{n=1}^{\infty} \frac{3n}{n^2-8}$

$$\frac{3n}{n^2-8} > \frac{3n}{n^2} = \frac{3}{n},$$

which diverges, so by the Comparison Test our series diverges.

(2) $\sum_{n=1}^{\infty} \frac{n+1}{n^3+n}$

$$\frac{n+1}{n^3+n} = \frac{n+1}{n^2(n+1)} = \frac{1}{n^2},$$

which converges by p-test.

(3) $\sum_{n=1}^{\infty} \frac{(n)(n+1)}{(n+2)(n+3)}$

$$\lim_{n\to\infty} \frac{n(n+1)}{(n+2)(n+3)} = \lim_{n\to\infty} \frac{n^2+n}{n^2+5n+6} \overset{H}{=} \lim_{n\to\infty} \frac{2n+1}{2n+5} \overset{H}{=} \lim_{n\to\infty} \frac{2}{2} \neq 0,$$

so by the Divergence Test this diverges.

(4) $\sum_{n=1}^{\infty} \frac{e^{-n}}{2n^2+3n+5}$

$$\frac{e^{-n}}{2n^2+3n+5} < e^{-n}$$

The right side is a geometric series with $r = \frac{1}{e}$ and thus converges, so by the Comparison Test our series converges.

(5) $\sum_{n=1}^{\infty} \frac{3n^3-7}{\sqrt{n^7+6n^2}}$

We want to compare to $\frac{3n^3}{\sqrt{n^7}} = \frac{3}{n^{0.5}}$, which is divergent because of the p-test. However, our series is less than this one, which means we won't be able to use the Comparison Test. In this case, we resort to the Limit Comparison Test.

$$\lim_{n\to\infty} \frac{\frac{3n^3-7}{\sqrt{n^7+6n^2}}}{\frac{3n^3}{\sqrt{n^7}}} = \lim_{n\to\infty} \frac{3n^3-7}{3n^3} \frac{\sqrt{n^7}}{\sqrt{n^7+6n^2}} = \lim_{n\to\infty} \left(1 - \frac{7}{3n^3}\right)\left(\sqrt{\frac{n^7}{n^7+6n^2}}\right) = 1$$

Thus, both are divergent.

(6) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^2 e^{-n}$

Knowing that the e^{-n} term will significantly dominate this series, we compare simply to e^{-n} with limit comparison to find

$$\lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^2 e^{-n}}{e^{-n}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2 = 1^2 = 1,$$

and because e^{-n} is a convergent geometric series, our series is convergent as well.

(7) $\sum_{n=1}^{\infty} \frac{1}{n!}$

$$\frac{1}{n!} = \frac{1}{(n)(n-1)(n-2)...(3)(2)(1)} = \frac{1}{n}\frac{1}{n-1}\frac{1}{n-2}\frac{1}{n-3}...\frac{1}{2}\frac{1}{1} < \frac{1}{2}\frac{1}{2}...\frac{1}{2} = \frac{1}{2^n}.$$

Thus, $\frac{1}{n!} < \frac{1}{2^n}$, and because the right side is a convergent Geometric Series, both are convergent by the Comparison test.

Bonus: We know that the Harmonic Series $H_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges. The Kempner Series is the series composed of the sum of the terms of the Harmonic Series that do not contain a 9. For example, the terms $\frac{1}{9}, \frac{1}{19}, \frac{1}{29}, \frac{1}{99}$, and $\frac{1}{397}$ and so on are all omitted from the sum. Prove that the Kempner Series converges using the following steps.

(1) Prove that the number of terms in the Kempner Series with n digits in the denominator equal to $8(9^{n-1})$.

For any n-digit number, the first digit can be any number from 1-8, as it can not be 9 by definition and it can not be 0 or else the number would not be an n-digit number. There are therefore 8 choices for the first digit. The rest of the digits can all take values from 0-8, which means there are 9 choices for each of these digits. Therefore, in total, there are $8(9)^{n-1}$ possible numbers of n digits where none of the digits are 9.

(2) Prove that any term with n digits in the denominator is less than $\frac{1}{10^{n-1}}$.

The smallest number with n digits is 10^{n-1}. Thus, any fraction with a 1 in the numerator and an n-digit number in the denominator will be less than (or equal to) $\frac{1}{10^{n-1}}$.

(3) Demonstrate that the Kempner Series follows the following inequality:

$$\sum_{n=1}^{\infty} K_n \leq \sum_{n=1}^{\infty} 8(9^{n-1})\frac{1}{10^{n-1}}$$

If we select a number n, we can see that there are $8(9^{n-1})$ numbers in the Kempner Series with n digits, and each one of them is less than $\frac{1}{10^{n-1}}$. Therefore, the sum of all of the terms in the Kempner Series with n-digits is less than $8(9^{n-1})\frac{1}{10^{n-1}}$, so if we add this up for all n, we can see that the total sum is less than $\sum_{n=1}^{\infty} 8(9^{n-1})\frac{1}{10^{n-1}}$.

(4) Use this inequality and the Comparison Test to determine that the Kempner Series converges, and find an upper bound for its sum.

Rearranging the sum we found, we note that

$$\sum_{n=1}^{\infty} 8(9^{n-1})\frac{1}{10^{n-1}} = 8\sum_{n=1}^{\infty} \frac{9^{n-1}}{10^{n-1}} = 8\sum_{n=0}^{\infty} \left(\frac{9}{10}\right)^n = 8(10) = 80,$$

so we can see that $\sum_{n=1}^{\infty} K_n \leq 80$ so it must converge to a number less than or equal to 80.