MATH 1B DISCUSSION WORKSHEET - 10/16/18

POWER SERIES AND FUNCTION REPRESENTATION ANSWERS

(1) Find the interval and radius of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{(x-6)^n}{n^n}.$$

Everything in this series is raised to the power of n, so we can use the root test!

$$\lim_{n\to\infty} \sqrt[n]{\frac{(x-6)^n}{n^n}} = \lim_{n\to\infty} \frac{(x-6)}{n} = 0$$

for any finite value of x, which means that it converges at all points! Thus, the radius is ∞, and the interval is $(-\infty, \infty)$.

(2) Find the interval and radius of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{2^n}{n}(3x-6)^n.$$

We can't use the root test for this one due to the n term, so we use the ratio test instead. Using the ratio test, we can see

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left| \frac{\frac{2^{n+1}(3x-6)^{n+1}}{n+1}}{\frac{2^n(3x-6)^n}{n}} \right| \\
&= \lim_{n\to\infty} \left|\frac{2^{n+1}}{2^n}\right| \left|\frac{(3x-6)^{n+1}}{(3x-6)^n}\right| \left|\frac{n}{n+1}\right| \\
&= \lim_{n\to\infty}(2) \cdot \lim_{n\to\infty}(3x-6) \cdot \lim_{n\to\infty} \frac{n}{n+1} \\
&= 2(3x-6)(1).
\end{aligned}$$

We want to find where this is less than one, because the ratio test tells us that the series is convergent when the absolute value of this ratio is less than one. Setting up the inequality, we find

$$\begin{aligned}
|2(3x-6)(1)| &< 1 \\
-1 < 6x - 12 &< 1 \\
\frac{-1}{6} < x - 2 &< \frac{1}{6} \\
\frac{11}{6} < x &< \frac{13}{6}.
\end{aligned}$$

Before we declare that this is the interval of convergence, we'll have to determine whether the bounds of the error make the series convergent or divergent. Plugging $x = \frac{11}{6}$ in gives

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{2^n}{n}(3x-6)^n &= \sum_{n=0}^{\infty} \frac{2^n}{n}(3(11/6)-6)^n \\ &= \sum_{n=0}^{\infty} \frac{2^n}{n}(5.5-6)^n \\ &= \sum_{n=0}^{\infty} \frac{2^n}{n}(-0.5)^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n}\end{aligned}$$

This is the alternating harmonic series, which is convergent, so $x = \frac{11}{6}$ gives a convergent series. On the other hand, plugging in $x = \frac{13}{6}$ gives

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{2^n}{n}(3x-6)^n &= \sum_{n=0}^{\infty} \frac{2^n}{n}(3(13/6)-6)^n \\ &= \sum_{n=0}^{\infty} \frac{2^n}{n}(6.5-6)^n \\ &= \sum_{n=0}^{\infty} \frac{2^n}{n}(0.5)^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n}\end{aligned}$$

This is the harmonic series, which we know diverges. Thus, $x = \frac{13}{6}$ is not within our radius of convergence. We can then see that our radius of convergence is $\frac{1}{6}$ and our final interval of convergence is

$$\boxed{I = \left[\frac{11}{6}, \frac{13}{6}\right)}.$$

(3) Find the interval and radius of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{1}{(-3)^{2+n}\ (n^2+1)}(4x-12)^n.$$

We'll approach this problem the same way as the last one. We know that the root test won't help us because some terms aren't taken to the power of n, so we will instead use the ratio test.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{1}{(-3)^{2+n+1}\ ((n+1)^2+1)}(4x-12)^{n+1}}{\frac{1}{(-3)^{2+n}\ (n^2+1)}(4x-12)^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{(-3)^{2+n}}{(-3)^{2+n+1}}\right|\left|\frac{n^2+1}{(n+1)^2+1}\right|\left|\frac{(4x-12)^{n+1}}{(4x-12)^n}\right| \\
&= \frac{1}{3}(1)(4x-12).
\end{aligned}$$

We are looking for when this has absolute value less than one, so by rearranging we can see

$$\begin{aligned}
|\frac{1}{3}(1)(4x-12)| &< 1 \\
-3 < 4x - 12 &< 3 \\
9 < 4x &< 15 \\
\frac{9}{4} < x &< \frac{15}{4}.
\end{aligned}$$

Now, we check the bounds. Plugging in $x = \frac{9}{4}$, we find

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{(-3)^{2+n}\ (n^2+1)}(4x-12)^n. &= \sum_{n=0}^{\infty} \frac{1}{(-3)^{2+n}\ (n^2+1)}(4\frac{9}{4}-12)^n \\
&= \sum_{n=0}^{\infty} \frac{1}{(-3)^{2+n}\ (n^2+1)}(-3)^n \\
&= \sum_{n=0}^{\infty} \frac{1}{9(n^2+1)} \leq \sum_{n=0}^{\infty} \frac{1}{n^2}
\end{aligned}$$

So by the comparison test, this series converges.

On the other hand, plugging in $x = \frac{15}{4}$, we find

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{(-3)^{2+n}\ (n^2+1)}(4x-12)^n. &= \sum_{n=0}^{\infty} \frac{1}{(-3)^{2+n}\ (n^2+1)}(4\frac{15}{4}-12)^n \\
&= \sum_{n=0}^{\infty} \frac{1}{(-3)^{2+n}\ (n^2+1)}(3)^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{9(n^2+1)},
\end{aligned}$$

which converges by the alternating series test. Thus, both bounds create convergent series as well, which means our final radius of convergence is $R = \frac{3}{4}$ and our interval is

$$\boxed{I = \left[\frac{9}{4}, \frac{15}{4}\right]}.$$

(4) Write the given functions by their power series and find the interval of convergence.

(a) We are looking to rewrite this in the form $a_1 \frac{1}{1-r}$. Taking the original fraction, we rewrite it to

$$\begin{aligned} \frac{3x^2}{4-2x} &= 3x^2 \frac{1}{4-2x} \\ &= 3x^2 \frac{1}{4\left(1-\frac{1}{2}x\right)} \\ &= \frac{3}{4}x^2 \frac{1}{1-\frac{1}{2}x}. \end{aligned}$$

From here, we use the fact that $\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$ to find

$$\begin{aligned} \frac{3}{4}x^2 \frac{1}{1-\frac{1}{2}x} &= \frac{3}{4}x^2 \sum_{n=0}^{\infty} \left(\frac{1}{2}x\right)^n \\ &= \frac{3}{4}x^2 \sum_{n=0}^{\infty} \frac{1}{2^n}x^n \\ &= \frac{3}{4} \sum_{n=0}^{\infty} \frac{1}{2^n}x^{n+2} \\ &= \frac{3}{4} \sum_{n=2}^{\infty} \frac{1}{2^{n-2}}x^n \\ &= \frac{3}{4} \sum_{n=2}^{\infty} \frac{4}{2^n}x^n \\ &= \boxed{\sum_{n=2}^{\infty} \frac{3}{2^n}x^n} \end{aligned}$$

(b) We note that $\int \frac{1}{(1+x)^2} = -\frac{1}{1+x} + C$. Thus, we can see that

$$\begin{aligned} \int \frac{1}{(1+x)^2} &= -\sum_{n=0}^{\infty} (-x)^n \\ \int \frac{1}{(1+x)^2} &= \sum_{n=0}^{\infty} (-1)^{n+1} x^n \\ \frac{d}{dx} \int \frac{1}{(1+x)^2} &= \frac{d}{dx} \sum_{n=0}^{\infty} (-1)^{n+1} x^n \\ \frac{1}{(1+x)^2} &= \sum_{n=1}^{\infty} (-1)^{n+1} n x^{n-1} \\ \frac{1}{(1+x)^2} &= \boxed{\sum_{n=0}^{\infty} (-1)^n (n+1) x^n} \end{aligned}$$

(5) Find the sum of the alternating harmonic series!

(a) Find the derivative of $\ln(1+x)$, and find its derivative's power series.
The derivative of $\ln(1+x)$ is $\frac{1}{1+x}$, which is equal to $\frac{1}{1-(-x)}$, so we can see that

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n.$$

(b) Use this to find the power series of $\ln(1+x)$.
Using the information from part (a), we can see that

$$\begin{aligned}
\frac{d}{dx}\ln(1+x) &= \sum_{n=0}^{\infty}(-1)^n x^n \\
\int \frac{d}{dx}\ln(1+x) &= \int \sum_{n=0}^{\infty}(-1)^n x^n \\
\ln(1+x) &= \sum_{n=0}^{\infty}(-1)^n \frac{1}{n+1} x^{n+1} \\
\ln(1+x) &= \sum_{n=1}^{\infty}(-1)^{n+1} \frac{1}{n} x^n
\end{aligned}$$

(c) What's the sum of the alternating harmonic series?
We know that the alternating harmonic series is of the form

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n},$$

and we can see that

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}x^n$$

so the two sums only differ in that the second sum has an x term. If we get rid of it by setting $x = 1$, we can see that

$$\begin{aligned}
\ln(1+x) &= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}x^n \\
\ln(1+1) &= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}(1)^n \\
\ln(2) &= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}
\end{aligned}$$

so the sum of the alternating harmonic series is $\boxed{\ln(2)}$.

(6) Note that e^x is a positive, nonzero function whose derivative is itself. Furthermore, the nth derivative of e^x is itself as well for any positive integer n. Find the power series representation of e^x by finding the only power series satisfying the properties listed above.

We are looking for a way to represent e^x as a power function:

$$e^x = \sum_{n=0}^{\infty} c_n x^n$$

for some values c_n. First, note that $e^0 = 1$, which means that

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} c_n x^n \\ e^0 &= \sum_{n=0}^{\infty} c_n (0)^n \\ 1 &= c_0 + c_1(0) + c_2(0) + ... \\ 1 &= c_0. \end{aligned}$$

From here, we know that $c_0 = 1$, which tells us not only that the first term of the original power series but also that every derivative of the power series still has first term 1. For any integer n, the term $\frac{x^n}{n!}$ has nth derivative of 1, which is important because it means that the sum must consist exactly of these terms in order for every single derivative to have first term 1. In particular, it means that

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + ... = \boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!}}.$$

(7) (Bonus) Find the power series representations for $\sin(x)$ and $\cos(x)$ by noting the following properties:

- $(\sin(x))'' = -\sin(x), (\cos(x))'' = -\cos(x)$.
- $\sin(x)$ is an odd function, while $\cos(x)$ is an even function.
- $(\sin(x))' = \cos(x)$, $(\cos(x))' = -\sin(x)$.

We first note that because cos is an even function, it only has even terms. Furthermore, because its second derivative is the negative of itself, we can see that it follows almost the same properties as e^x, except that it is alternating. Coupled with the fact that $\cos(0) = 1$, We can then see that $\cos(x)$ should look like

$$\boxed{\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}}$$

in order to maintain these properties. We can then see that the integral of this function is

$$\begin{aligned} \int \cos(x) &= \int \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...\right) dx \\ \sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... \\ \sin(x) &= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}} \end{aligned}$$

(8) (Extra bonus for overachievers) Find the value of

$$\frac{1}{\sqrt{2}} + \frac{4}{2} + \frac{9}{2\sqrt{2}} + \frac{16}{4} + \cdots$$

Let's start by replacing $\frac{1}{\sqrt{2}}$ with x for simplicity; we can replace it at the end. We then call the sum S. This summation then becomes $S = \sum_{n=1}^{\infty}(n^2)(x^n) = 1^2x^1 + 2^2x^2 + 3^2x^3 + 4^2x^4...$ We then notice that the common difference between square numbers is the odd numbers (e.g. $2^2 - 1^2 = 3, 3^2 - 2^2 = 5, 4^2 - 3^2 = 7$, etc.), which we want to obtain, so with a bit of algebraic manipulation we find that

$$\begin{aligned} S &= 1^2x^1 + 2^2x^2 + 3^2x^3 + 4^2x^4... \\ -xS &= \qquad - 1^2x^2 - 2^2x^3 - 3^2x^4... \\ (1-x)S = S - xS &= 1x^1 + 3x^2 + 5x^3 + 7x^4... \end{aligned}$$

Now that the terms are increasing by a factor of two, we want to do manipulate it one more time to make the coefficients constant: this makes the equation even easier to work with. We notice that the common difference between the terms is now 2, and we do the same process again:

$$\begin{aligned} (1-x)S &= 1x^1 + 3x^2 + 5x^3 + 7x^4... \\ -x(1-x)S &= \qquad - 1x^2 - 3x^3 - 5x^4... \\ (1-x)(1-x)S &= 1x^1 + 2x^2 + 2x^3 + 2x^4... \\ (1-x)^2S &= x + 2(x^2 + x^3 + x^4...) \end{aligned}$$

The only part of the equation left to simplify before we can plug x back in to solve for S is the infinite series $x^2 + x^3 + x^4....$ Once again, we want to simplify the coefficients; but now that their common difference is zero, the simplification nullifies the series altogther, which is extremely helpful for computation. Let us call $s = x^2 + x^3 + x^4...$, and we can simplify it like this:

$$\begin{aligned} s &= x^2 + x^3 + x^4... \\ -xs &= \qquad - x^3 - x^4... \\ x - xs = (1-x)s &= x^2 \\ s &= \frac{x^2}{1-x} \end{aligned}$$

We are now ready to substitute $\frac{1}{\sqrt{2}}$ back in for x, and solve for S.

$$\begin{aligned}
(1-x)^2 S &= x + 2(x^2 + x^3 + x^4 ...) \\
(1-x)^2 S &= x + 2s \\
(1 - \frac{1}{\sqrt{2}})^2 S &= \frac{1}{\sqrt{2}} + (2)\frac{(\frac{1}{\sqrt{2}})^2}{1 - \frac{1}{\sqrt{2}}} \\
(\frac{\sqrt{2}-1}{\sqrt{2}})^2 S &= \frac{\sqrt{2}}{2} + (2)\frac{\frac{1}{2}}{\frac{\sqrt{2}-1}{\sqrt{2}}} \\
\frac{2 - 2\sqrt{2} + 1}{2} S &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{\sqrt{2}-1} \\
\frac{3 - 2\sqrt{2}}{2} S &= \frac{\sqrt{2}}{2} + \frac{(\sqrt{2})(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)} \\
\frac{3 - 2\sqrt{2}}{2} S &= \frac{\sqrt{2}}{2} + \frac{2+\sqrt{2}}{1} \\
(3 - 2\sqrt{2}) S &= \sqrt{2} + 2(2 + \sqrt{2}) \\
S &= \frac{4 + 3\sqrt{2}}{3 - 2\sqrt{2}} = \frac{(4+3\sqrt{2})(3+2\sqrt{2})}{(3-2\sqrt{2})(3+2\sqrt{2})} \\
S &= \frac{12 + 17\sqrt{2} + 12}{9 - 8} \\
S &= \boxed{24 + 17\sqrt{2}}
\end{aligned}$$