MATH 1B DISCUSSION WORKSHEET - 10/23/18

MIDTERM 2 REVIEW

1. Factorial Algebra

Factorials have started to show up a lot in the second half of this chapter! They're a great sign that you're going to want to be using the Ratio or Comparison Tests, and they also show up in the formula for a Taylor Expansion. For this reason, it's extremely important to know how to work with them. Writing them out is almost always helpful!

(1) Rewrite $(n+2)(n+1)(n!)$ as a single factorial.

$(n+2)!$

(2) Consider the fraction $\frac{(2n)!}{n!}$.

(a) Write out the numerator's product. How many terms are in the numerator?

$(2n)(2n-1)(2n-2)\cdots(2)(1)$

There are $2n$ terms

(b) Write out the denominator's product. How many terms are in the denominator?

$(n)(n-1)(n-2)\cdots$

There are n terms.

(c) Is there overlap between the terms above and below?

Yes, on the terms $(n)(n-1)(n-2)\cdots(2)(1)$.

(d) What does the fraction simplify to?

$$\frac{(2n)(2n-1)(2n-2)\cdots(n+1)(n)(n-1)\cdots(2)(1)}{n(n-1)(n-2)(n-3)\cdots(2)(1)} = \boxed{(2n)(2n-1)\cdots(n+1)}$$

(3) Consider the sequence given by $a_n = (n+4)!$.

(a) Write out the products from a_3 and a_4 (don't calculate them though). How many terms are in each of these products?

$a_3 = 7! = 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1$

$a_4 = 8! = 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1$

(b) Find $\frac{a_4}{a_3}$.

$$\frac{a_4}{a_3} = \frac{8\cdot 7\cdot 6\cdots}{7\cdot 6\cdots} = \boxed{8}$$

(c) Find and simplify $\frac{a_{n+1}}{a_n}$.

$a_{n+1} = (n+5)!$

$$\frac{a_{n+1}}{a_n} = \frac{(n+5)(n+4)\cdots(2)(1)}{(n+4)(n+3)\cdots(2)(1)} = \boxed{n+5}$$

(4) Consider the sequence given by $a_n = (3n+2)!$.
   (a) Write out the products from a_3 and a_4 (don't calculate them though). How many terms are in each of these products?

   $a_3 = (9+2)! = 11 \cdot 10 \cdot 9 \cdot 8 \cdots \Rightarrow 11$ terms

   $a_4 = (14)! = 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdots \Rightarrow 14$ terms

   (b) Find $\frac{a_4}{a_3}$.

   $$\frac{14!}{11!} = \frac{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdots}{11 \cdot 10 \cdot 9 \cdot 8 \cdots} = \boxed{14 \cdot 13 \cdot 12}$$

   (c) Find and simplify $\frac{a_{n+1}}{a_n}$.

   $$\frac{(3(n+1)+2)!}{(3n+2)!} = \frac{(3n+5)!}{(3n+2)!} = \frac{(3n+5)(3n+4)(3n+3)(3n+2)\cdots}{(3n+2)(3n+1)\cdots} = \boxed{(3n+5)(3n+4)(3n+3)}$$

(5) Follow the steps below to determine whether $(2n)!$ or $(n!)^2$ is larger.
   (a) Write out the products for $n = 3$.

   $(2n)! = 6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

   $(n!)^2 = 3! \cdot 3! = 3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1$

   (b) How many terms are in each product, in terms of n?

   There are $2n$ terms in each one.

   (c) Which one is larger?

   $(2n)!$, because its terms are larger.

(6) Follow the steps below to determine $\lim_{n\to\infty} \frac{2^n}{n!}$
   (a) Both of these are products as well! How many terms are in each product?

   2^n has n terms. $n!$ has n terms as well.

   (b) Try writing the fraction out for $n = 5$.

   $$\frac{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

   (c) Is the numerator larger or smaller than the denominator?

   Smaller.

   (d) As n increases, does the difference between the two widen?

   Yes.

   (e) What's the limit as n approaches infinity?

   0.

2. Power Series Manipulation

Becoming comfortable manipulating Power Series is far more about practice than it is about rules. Once again, an important tool you should be using is writing anything out. Sigma notation can definitely become confusing, but rewriting it as a polynomial will be much easier for you to work with because you should be pretty comfortable with polynomials by now.

2.1. Using the $\frac{1}{1-x}$ power series.

(1) Write the sigma notation expansion for the following. If you get stuck, try writing out some terms!

(a) $\frac{1}{1-x}$ $\boxed{\sum_{n=0}^{\infty} x^n}$

(b) $\frac{1}{1+2x}$

$$\sum_{n=0}^{\infty} (-2x)^n = \boxed{\sum_{n=0}^{\infty} (-1)^n 2^n x^n}$$

(c) $\frac{1}{3-x}$

$$\frac{1}{3-x} = \frac{1}{3}\frac{1}{1-\frac{x}{3}} = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = \boxed{\sum_{n=0}^{\infty}\frac{1}{3^{n+1}}x^n}$$

(d) $\frac{x^4}{1+x^3}$

$$x^4\sum_{n=0}^{\infty}(-x^3)^n = x^4\sum_{n=0}^{\infty}(-1)^n x^{3n} = \boxed{\sum_{n=0}^{\infty}(-1)^n x^{3n+4}}$$

(e) $\frac{x^2}{4+6x^5}$

$$x^2\left(\frac{1}{4+6x^5}\right) = x^2\left(\frac{1}{4}\right)\left(\frac{1}{1+\frac{3}{2}x^5}\right) = \frac{x^2}{4}\sum_{n=0}^{\infty}\left(\frac{-3}{2}x^5\right)^n = \boxed{\sum_{n=0}^{\infty}(-1)^n\frac{3^n}{2^{n+2}}\cdot x^{5n+2}}$$

(2) Using the fact that $\frac{d}{dx}\ln(1+x) = \frac{1}{1+x}$ and $\cancel{\frac{d}{dx}}\tan^{-1}(x) = \int\frac{1}{1+x^2}$, write the sigma notation for each of the following:

(a) $\ln(1+x)$

$$\frac{1}{1+y} = \sum(-x)^n \qquad \int\frac{1}{1+y} = \int\sum(-1)^n x^n = \boxed{\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}}$$

(b) $\tan^{-1}(x)$

$$\frac{1}{1+y^2} = \sum_{n=0}^{\infty}(-y^2)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n} \qquad \int\frac{1}{1+y^2} = \int\sum_{n=0}^{\infty}(-1)^n x^{2n} = \boxed{\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}}$$

(c) $\ln(3+x^2)$

$$\ln(3+x^2) = \int 2x\frac{1}{3+y^2} = \int\frac{2}{3}x\left(\frac{1}{1+\frac{x^2}{3}}\right) = \int\frac{2}{3}x\sum-\left(\frac{x^2}{3}\right)^n = \int\sum(-1)^n 2\cdot\frac{x^{2n+1}}{3^{n+1}}$$

$$= \boxed{\sum(-1)^n\frac{x^{2n+2}}{(n+1)3^{n+1}}}$$

(d) $x\tan^{-1}(2x^2)$

$$= x\sum_{x=0}^{\infty}(-1)^n\frac{(2x^2)^{2n+1}}{2n+1} = \boxed{\sum_{x=0}^{\infty}(-1)^n\frac{2^{2n+1}}{2n+1}\cdot x^{4n+3}}$$

2.2. Dealing with the sin and cos power series.

(1) Write the Maclaurin series for $\cos(x)$, in sigma notation as well as writing out the first four terms.

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

(2) Write the Maclaurin series for $\cos(x^2)$, in sigma notation as well as writing out the first four terms.

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{(2n)!} = 1 - \frac{x^4}{2} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \cdots$$

(3) Differentiate the sum both by differentiating each of the four terms as well as by differentiating within the sigma notation. Ensure that they are the same (check the bounds!).

$$\sum_{n=1} (-1)^n \frac{4n\, x^{4n-1}}{(2n)!} = -\frac{4x^3}{2} + \frac{8x^7}{4!} - \frac{12x^{11}}{6!}$$

(4) Write the Maclaurin Series for $x\sin(x^2)$.

$$\sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} \qquad \boxed{x\sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(2n+1)!}}$$

(5) Show that $-2x\sin(x^2)$ is equal to the derivative of $\cos(x^2)$.

$$(-2)\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(2n+1)!} = (-2)\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{4n-1}}{(2n-1)!} = \sum_{n=1}^{\infty} (-1)^n \frac{2x^{4n-1}}{(2n-1)!} = \sum_{n=1}^{\infty} (-1)^n \frac{4n}{2n(2n-1)!} x^{4n-1} \checkmark$$

2.3. Identifying Maclaurin Series.
Convert each of the following into a Maclaurin Series.

(1) $x\cos(2x)$

$$\cos(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} \qquad \boxed{x\cos(2x) = \sum_{n=0}^{\infty} (-1)^n \frac{(2)^{2n} x^{2n+1}}{(2n)!}} \qquad = \sum_{n=1}^{\infty} (-1)^n \frac{4n x^{4n-1}}{(2n)!}$$

(2) e^{-x^2}

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}}$$

(3) $x\cos\left(\frac{1}{2}x^2\right)$

$$x\sum_{n=0}^{\infty} (-1)^n \frac{(\frac{1}{2}x^2)^{2n}}{(2n)!} = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{2^{2n}(2n)!}}$$

(4) $x^3(1+x)^2$

$$x^3 \sum \binom{2}{n} x^n = \boxed{\sum_{n=0}^{\infty} \binom{2}{n} x^{n+3}}$$

Determine the function modeled by each of the following Maclaurin Series.

(1) $1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \ldots$ $\boxed{f(x) = e^{-x^2}}$

(2) $\sum_{n=0}^{\infty} (-1)^n \frac{3(9^n)x^{2n+1}}{2n+1}$ $\quad 3(9^n) = 3^{2n+1}$, so this is $\boxed{f(x) = \sin(3x)}$

(3) $1 + x - \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!} - \ldots$

$\boxed{f(x) = \sin(x) + \cos(x)}$

3. Absolute and Conditional Convergence

First, let's determine how to idenitfy when a series is alternating. For each of the following sequences, list the first few terms.

(1) $a_n = (-1)^n$

$1, -1, 1, -1, \ldots$

(2) $a_n = \cos(\pi n)$

$1, -1, 1, -1, \ldots$

(3) $a_n = \sin((n + \frac{1}{2})\pi)$

$1, -1, 1, -1, \ldots$

If any of these are in the series, then we know it's alternating!

Now that we know how to identify alternating series, let's make a step by step procedure for how to determine whether it is Absolutely Convergent, Conditionally Convergent, or Divergent.

(1) Step 1: Perform Alternating Series Test!
How do we perform the Alternating Series Test?

If $|a_{n+1}| > |a_n|$ for all n and $\lim_{n\to\infty} a_n = 0$, then the alternating series converges.

Does it pass AST?
- YES: Proceed to step 2.
- NO: The serie is divergent.

(2) Step 2: Perform a different test on the absolute value of the series! What are the most common tests we use with respect to alternating series?

Ratio, Root.

Does it converge when you use those tests?
- YES: Absolutely Convergent.
- NO: Conditionally Convergent.

4. Telescoping Series

Telescoping Series are series whose terms cancel out with themselves. It's unlikely that they will show up on the test, but they are definitely handy sometimes and are worth talking about. This is a prime example of a series that isn't geometric but still has a sum that isn't too hard to calculate.

Let's walk through an example. Let's say we are trying to find the sum $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. At face value, it's definitely not easy to find the sum of, but let's find a way around it.

(1) Using partial fraction decomposition, rewrite $\frac{1}{n(n+1)}$ as the difference of two fractions.

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

(2) Write out the first few terms of the sum using the partial fraction decomposition. What do you notice about the sum? Does anything cancel out nicely?

$$\left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots$$

(3) What is the final sum?

$$\boxed{1}$$

This is a great example, but there are actually many examples of telescoping sums! Here are a few practice questions:

(1) Find $\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n})$, and then find $\sum_{n=0}^{\infty} (\sqrt{n+1} - \sqrt{n})$. $\longrightarrow$ $\sum_{n=0}^{\infty} \sqrt{n+1} - \sqrt{n}$

$$= (\sqrt{1} - \sqrt{0}) + (\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) + (\sqrt{4} - \sqrt{3}) + \cdots$$

$$= \sqrt{\infty} = \boxed{\text{Divergent!}}$$

$$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = \lim_{n\to\infty} \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} = 1.$$

(2) Find $\sum_{n=1}^{\infty} \frac{1}{n^2 + 3n + 2}$.

$$\frac{1}{n^2+3n+2} = \frac{1}{n+1} - \frac{1}{n+2} \qquad \sum_{n=1}^{\infty} \frac{1}{n+1} - \frac{1}{n+1} = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) \cdots = \boxed{\frac{1}{2}}$$

(3) Find $\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$.

$$\frac{1}{(2n+1)(2n-1)} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) \qquad \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) = \frac{1}{2}\left[\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots\right]$$

$$= \boxed{\frac{1}{2}}$$

5. Extra pages for two problems that are significantly harder than you'll find on the midterm but are still interesting problems that can be solved using only material we've learned in class! (From *Real Mathematical Analysis*) by Pugh.

(1) Prove that if the terms of a sequence decrease monotonically ($a_1 \geq a_2 \geq ...$) and converge to 0 then the series $\sum_{n=1}^{\infty} a_k$ converges *if and only if* the associated dyadic series

$$a_1 + 2a_2 + 4a_4 + 8a_8 + ... = \sum_{n=1}^{\infty} 2^k a_{2^k}$$

converges. [Hint: How can we compare the dyadic series to the original one?]

Note that "if and only if" means that you'll have to prove two statements:

(a) If the original series converges, then the dyadic one does too.

(b) If the dyadic series converges, then the original one does too.

Seeing that if the dyadic series' convergence implies that the original series converges as well is easy. If we write the dyadic series as

$$a_1 + a_2 + a_2 + a_4 + a_4 + ...$$

and compare it to the original sequence

$$a_1 + a_2 + a_3 + a_4 + a_5 + ...$$

we can see that each term in the dyadic sequence is greater than or equal to its corresponding term in the original sequence. Therefore, by comparison, if the dyadic sequence converges, then the original one does too.

On the other hand, let's take the dyadic series and divide it by two. Then we see that

$$\frac{1}{2} \sum 2^k a_{2^k} = \frac{1}{2} a_1 + a_2 + 2a_4 + 4a_8.$$

If we try to compare again, we rewrite the halved dyadic series as

$$0.5a_1 + a_2 + a_4 + a_4 + a_8 + ...$$

and compare once again to the original sequence

$$a_1 + a_2 + a_3 + a_4 + a_5 + ...$$

we can see that each term in the dyadic sequence is less than or equal to its corresponding term in the original sequence. The terms at indices 2^n are equal, but at any index $i = 2^n + m$ where n is the largest integer such that $i > 2^n$, we find that in the dyadic sequence this corresponds to $a_{2^{n+1}}$ but the original sequence it corresponds to simply a_i, and we know that $i = 2^n + m < 2^{n+1}$ so $a_i > a_{2^{n+1}}$, which means the dyadic sequence is smaller. Therefore, we know that 2 times the original sequence is greater than its dyadic sequence, so if the original sequence, then double the original sequence must converge as well, which by comparison test tells us that the dyadic sequence converges.

In essence, we have shown that the dyadic series is greater than the original series but less than a constant multiple of the original series, so it is clear that the long-term behavior of one is directly associated with the long-term behavior of the other.

(2) An **infinite product** is an expression $\prod_{n=1}^{\infty} c_k$ where $c_k > 0$. It's the equivalent of an infinite sum, but the terms are multiplied together instead of being added together. The n^{th} **partial product** is $C_n = (c_1)(c_2)...(c_n)$. If C_n converges to a limit $C \neq 0$ then we say that the product converges to C. Denote $c_k = 1 + a_k$. If each $a_k \geq 0$, prove that $\sum_{n=1}^{\infty} a_k$ converges *if and only if* $\prod_{n=1}^{\infty} c_k$ converges. [Hint: Take logarithms.]

First, we take the sums at face value. Let's look at the comparison between the sums and products. For any a_1 and a_2, we can see that

$$(1 + a_1)(1 + a_2) = 1 + a_1 + a_2 + a_1 a_2 > a_1 + a_2$$

for any positive or negative a_1 and a_2, because in either case $a_1 a_2$ is positive so it must be greater. As a result, if we apply this to the sequences, we can see that the second partial sum is less than or equal to the second partial product. Furthermore, for any n^{th} partial product, we can apply the same principle to see that the $(n+1)^{\text{th}}$ partial product is greater than the $(n+1)^{\text{th}}$ partial sum, regardless of whether a_n is positive or negative, as long as each term is the same sign. As a result, we can tell through the comparison test that if the sum diverges then the product must diverge too, and if the product converges then the sum must converge too.

What's left, then, is to show that the divergence of the product implies the divergence of the sum, and the convergence of the sum implies the convergence of the product. Taking the logarithm of the infinite product, we can see that $\log\left(\prod c_k\right) = \sum \log(c_k) = \sum \log(a_k + 1)$. If we compare a_k and $\log(a_k + 1)$, we know that $a_k \geq \log(a_k + 1) \implies e^{a_k} \geq a_k + 1$ because e^x is monotonically increasing. Now, we know that if $a_k = 0$, then $e^0 = 0 + 1 = 1$. For all positive x, the derivative of e^x with respect to x is greater than 1, but the derivative of $x + 1$ with respect to x is 1, so we know that e^x grows faster than $x + 1$ and the two are equal at $x = 0$, so for all positive x we know that $e^{a_k} > a_k + 1$. Furthermore, for negative x, we know that $e^x < 1$ so the derivative of e^x is smaller than the derivative of $x + 1$ for all x implying that when $x < 0$ the value of $x + 1$ is less than the value of e^x. Therefore, for all x, we know that $e^x > x + 1$, which allows us to assert that $a_k > \log(a_k + 1)$ for all a_k.

As a result, $a_k \geq 0$, then the comparison test tells us that the infinite sum's convergence implies the infinite product's convergence, and if $a_k \leq 0$, then the infinite product's convergence implies the infinite sum's convergence. (It actually tells us about the convergence of the log of the infinite product, but because $c_k > 0$ we know that the product is positive, in which case log is continuous so the convergence of the log of the infinite product to some L implies the convergence of the infinite product to e^L.)

We have therefore shown that $\sum a_k$ converges if and only if $\prod c_k$ converges.