MATH 1B DISCUSSION WORKSHEET - 10/23/18

MIDTERM 2 REVIEW

1. Factorial Algebra

Factorials have started to show up a lot in the second half of this chapter! They're a great sign that you're going to want to be using the Ratio or Comparison Tests, and they also show up in the formula for a Taylor Expansion. For this reason, it's extremely important to know how to work with them. Writing them out is almost always helpful!

(1) Rewrite $(n+2)(n+1)(n!)$ as a single factorial.

(2) Consider the fraction $\frac{(2n)!}{n!}$.
   (a) Write out the numerator's product. How many terms are in the numerator?
   (b) Write out the denominator's product. How many terms are in the denominator?
   (c) Is there overlap between the terms above and below?
   (d) What does the fraction simplify to?

(3) Consider the sequence given by $a_n = (n+4)!$.
   (a) Write out the products from a_3 and a_4 (don't calculate them though). How many terms are in each of these products?
   (b) Find $\frac{a_4}{a_3}$.
   (c) Find and simplify $\frac{a_{n+1}}{a_n}$.

(4) Consider the sequence given by $a_n = (3n+2)!$.
   (a) Write out the products from a_3 and a_4 (don't calculate them though). How many terms are in each of these products?
   (b) Find $\frac{a_4}{a_3}$.
   (c) Find and simplify $\frac{a_{n+1}}{a_n}$.

(5) Follow the steps below to determine whether $(2n)!$ or $(n!)^2$ is larger.
   (a) Write out the products for $n = 3$.
   (b) How many terms are in each product, in terms of n?
   (c) Which one is larger?

(6) Follow the steps below to determine $\lim_{n\to\infty} \frac{2^n}{n!}$
   (a) Both of these are products as well! How many terms are in each product?
   (b) Try writing the fraction out for $n = 5$.
   (c) Is the numerator larger or smaller than the denominator?
   (d) As n increases, does the difference between the two widen?
   (e) What's the limit as n approaches infinity?

2. Power Series Manipulation

Becoming comfortable manipulating Power Series is far more about practice than it is about rules. Once again, an important tool you should be using is writing anything out. Sigma notation can definitely become confusing, but rewriting it as a polynomial will be much easier for you to work with because you should be pretty comfortable with polynomials by now.

2.1. Using the $\frac{1}{1-x}$ power series.

(1) Write the sigma notation expansion for the following. If you get stuck, try writing out some terms!

(a) $\dfrac{1}{1-x}$

(b) $\dfrac{1}{1+2x}$

(c) $\dfrac{1}{3-x}$

(d) $\dfrac{x^4}{1+x^3}$

(e) $\dfrac{x^2}{4+6x^5}$

(2) Using the fact that $\frac{d}{dx}\ln(1+x) = \frac{1}{1+x}$ and $\tan^{-1}(x) = \int \frac{1}{1+x^2}$, write the sigma notation for each of the following:

(a) $\ln(1+x)$

(b) $\tan^{-1}(x)$

(c) $\ln(3+x^2)$

(d) $x\tan^{-1}(2x^2)$

2.2. **Dealing with the** $\sin$ **and** $\cos$ **power series.**

(1) Write the Maclaurin series for $\cos(x)$, in sigma notation as well as writing out the first four terms.

(2) Write the Maclaurin series for $\cos(x^2)$, in sigma notation as well as writing out the first four terms.

(3) Differentiate the sum both by differentiating each of the four terms as well as by differentiating within the sigma notation. Ensure that they are the same (check the bounds!).

(4) Write the Maclaurin Series for $x\sin(x^2)$.

(5) Show that $-2x\sin(x^2)$ is equal to the derivative of $\cos(x^2)$.

2.3. **Identifying Maclaurin Series.** Convert each of the following into a Maclaurin Series.

(1) $x\cos(2x)$

(2) e^{-x^2}

(3) $x\cos\left(\frac{1}{2}x^2\right)$

(4) $x^3(1+x)^2$

Determine the function modeled by each of the following Maclaurin Series.

(1) $1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \ldots$

(2) $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{3(9^n)x^{2n+1}}{2n+1}$

(3) $1 + x - \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!} - \ldots$

3. Absolute and Conditional Convergence

First, let's determine how to idenitfy when a series is alternating. For each of the following sequences, list the first few terms.

(1) $a_n = (-1)^n$

(2) $a_n = \cos(\pi n)$

(3) $a_n = \sin((n + \frac{1}{2})\pi)$

If any of these are in the series, then we know it's alternating!

Now that we know how to identify alternating series, let's make a step by step procedure for how to determine whether it is Absolutely Convergent, Conditionally Convergent, or Divergent.

(1) Step 1: Perform Alternating Series Test!
How do we perform the Alternating Series Test?

Does it pass AST?
- YES: Proceed to step 2.
- NO: ____________________

(2) Step 2: Perform a different test on the absolute value of the series! What are the most common tests we use with respect to alternating series?

Does it converge when you use those tests?
- YES: ____________________
- NO: ____________________

4. Telescoping Series

Telescoping Series are series whose terms cancel out with themselves. It's unlikely that they will show up on the test, but they are definitely handy sometimes and are worth talking about. This is a prime example of a series that isn't geometric but still has a sum that isn't too hard to calculate.

Let's walk through an example. Let's say we are trying to find the sum $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. At face value, it's definitely not easy to find the sum of, but let's find a way around it.

(1) Using partial fraction decomposition, rewrite $\frac{1}{n(n+1)}$ as the difference of two fractions.

(2) Write out the first few terms of the sum using the partial fraction decomposition. What do you notice about the sum? Does anything cancel out nicely?

(3) What is the final sum?

This is a great example, but there are actually many examples of telescoping sums! Here are a few practice questions:

(1) Find $\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n})$, and then find $\displaystyle\sum_{n=0}^{\infty}(\sqrt{n+1}-\sqrt{n})$.

(2) Find $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+3n+2}$.

(3) Find $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)}$.

5. Extra pages for two problems that are significantly harder than you'll find on the midterm but are still interesting problems that can be solved using only material we've learned in class! (From *Real Mathematical Analysis*) by Pugh.

(1) Prove that if the terms of a sequence decrease monotonically ($a_1 \geq a_2 \geq \ldots$) and converge to 0 then the series $\sum_{n=1}^{\infty} a_k$ converges *if and only if* the associated dyadic series

$$a_1 + 2a_2 + 4a_4 + 8a_8 + \ldots = \sum_{n=1}^{\infty} 2^k a_{2^k}$$

converges. [Hint: How can we compare the dyadic series to the original one?]

Note that "if and only if" means that you'll have to prove two statements:

(a) If the original series converges, then the dyadic one does too.
(b) If the dyadic series converges, then the original one does too.

(2) An **infinite product** is an expression $\prod_{n=1}^{\infty} c_k$ where $c_k > 0$. It's the equivalent of an infinite sum, but the terms are multiplied together instead of being added together. The n^{th} **partial product** is $C_n = (c_1)(c_2)...(c_n)$. If C_n converges to a limit $C \neq 0$ then we say that the product converges to C. Denote $c_k = 1 + a_k$. If each $a_k \geq 0$, prove that $\sum_{n=1}^{\infty} a_k$ converges *if and only if* $\prod_{n=1}^{\infty} c_k$ converges. [Hint: Take logarithms.]