MATH 1B MIDTERM 2 MOCK EXAM

(1) Use the Test for Divergence to demonstrate that

$$\lim_{n\to\infty} \frac{(2n)^{2n}}{(3n)!} = 0.$$

The test for Divergence states that if $\sum_{n=0}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. Therefore, to prove that $\lim_{n\to\infty} \frac{2n^{2n}}{(3n)!} = 0$, we must show that $\sum_{n=0}^{\infty} \frac{2n^{2n}}{(3n)!}$ converges.

We will do this through the ratio test. Note that

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(2(n+1))^{2(n+1)}/(3(n+1))!}{2n^{2n}/(3n)!} \right|$$

$$= \lim_{n\to\infty} \left[\left(\frac{(2n+2)^{2n+2}}{2n^{2n}} \right) \left(\frac{(3n)!}{(3n+3)!} \right) \right]$$

$$= \lim_{n\to\infty} \left[(2n+2)^2 \left(\frac{2n+2}{2n} \right)^{2n} \frac{1}{(3n+3)(3n+2)(3n+1)} \right]$$

$$= \lim_{n\to\infty} \left[\frac{(2n+2)^2}{(3n+3)(3n+2)(3n+1)} \cdot \left(\left(1 + \frac{1}{n} \right)^n \right)^2 \right]$$

$$= \lim_{n\to\infty} \left[\frac{(2n+2)^2}{(3n+3)(3n+2)(3n+1)} \cdot e^2 \right]$$

The numerator is a second-degree polynomial, while the denominator is a third-degree polynomial, so the limit as $n\to\infty$ is 0. Thus, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$, so by the ratio test, $\sum_{n=0}^{\infty} \frac{(2n)^{2n}}{(3n)!}$ converges. By the Test for Divergence, we know then that $\boxed{\lim_{n\to\infty} \frac{2n^{2n}}{(3n)!} = 0}$ as desired.

(2) Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n^2+e^n}{1+(2n)^n}$$

We can compare this series to $\sum_{n=1}^{\infty} \frac{e^n}{(2n)^n}$. First, we should determine whether our new series converges or diverges.

Because both terms are raised to the power of n, we use the root test.

$$\lim_{n\to\infty} \sqrt[n]{\frac{e^n}{(2n)^n}} = \lim_{n\to\infty} \frac{e}{2n} = \frac{e}{\infty} = 0 < 1,$$

so by the root test, $\sum_{n=1}^{\infty} \frac{e^n}{(2n)^n}$ converges.

Now, we can see that $\frac{n^2+e^n}{1+(2n)^n} > \frac{e^n}{(2n)^n}$, so the comparison test doesn't help. Instead, we will use the Limit Comparison Test.

$$\lim_{n\to\infty} \frac{n^2+e^n/1+(2n)^n}{e^n/(2n)^n} = \lim_{n\to\infty} \left(\frac{n^2+e^n}{e^n}\right)\left(\frac{(2n)^n}{1+(2n)^n}\right) = \lim_{n\to\infty}\left(\frac{n^2+e^n}{e^n}\right)$$

$$\overset{L'H}{=} \lim_{n\to\infty}\left(\frac{2n+e^n}{e^n}\right) \overset{L'H}{=} \left(\frac{2+e^n}{e^n}\right) \overset{L'H}{=} \left(\frac{e^n}{e^n}\right) = 1,$$

and because $1 > 0$, our two series have the same behavior.

We have shown that $\sum_{n=1}^{\infty} \frac{e^n}{(2n)^n}$ converges, so the series

$\sum_{n=1}^{\infty} \frac{n^2+e^n}{1+(2n)^n}$ $\boxed{\text{converges}}$ as well.

(3) Determine whether the following series converges absolutely, converges conditionally, or diverges.

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln(\sqrt{n})}$$

Note that n, $\ln(n)$, and $\sqrt{n}$ are all increasing functions, so $|a_{n+1}| < |a_n|$ because the denominator increases with each term. Furthermore, $\lim_{n\to\infty} \frac{1}{n\ln(\sqrt{n})} = \frac{1}{\infty} = 0$, so by the Alternating Series Test, the alternating series converges. Now, we must determine whether $\sum_{n=2}^{\infty} \frac{1}{n\ln(\sqrt{n})}$ converges. We can do this with the integral test because the series is positive, decreasing, and continuous on n from $[2, \infty)$. We can do this multiple ways:

Method 1

$\ln(\sqrt{n}) = \ln(n^{1/2}) = \frac{1}{2}\ln(n)$, so

$$\int_2^{\infty} \frac{1}{n\ln(\sqrt{n})}dn = \frac{1}{2}\int_2^{\infty} \frac{1}{n\ln(n)}dn$$

$u = \ln(n)$

$du = \frac{1}{n}dn$

$$\lim_{t\to\infty} \frac{1}{2}\int_2^{t} \frac{1}{u}du$$

$$\lim_{t\to\infty} \frac{1}{2}\ln(u)\Big|_2^t$$

$$\lim_{t\to\infty} \frac{1}{2}\ln(\ln(u))\Big|_2^t$$

$\lim_{t\to\infty} \frac{1}{2}\ln(\ln(t)) = \infty$, so the series is divergent.

Method 2

$$\int_2^{\infty} \frac{1}{n\ln(\sqrt{n})}dn = \lim_{t\to\infty} \int_2^{t} \frac{1}{n\ln(\sqrt{n})}dn$$

$u = \ln(\sqrt{n})$

$du = \frac{1}{\sqrt{n}} \cdot \frac{1}{2\sqrt{n}} = \frac{1}{2n}dn$

$$= \lim_{t\to\infty} \int_2^{t} 2 \cdot \frac{1}{u}du$$

$$= \lim_{t\to\infty} 2\ln(u)\Big|_2^t$$

$$= \lim_{t\to\infty} 2\ln(\ln(\sqrt{n}))\Big|_2^t$$

$\lim_{t\to\infty} 2\ln(\ln(\sqrt{t})) = \infty$, so the series is divergent.

Either way, the non-alternating series is divergent, so we know that the alternating series is $\boxed{\text{conditionally convergent}}$.

(4) Find the first three nonzero terms of the Taylor Series approximation for

$$f(x) = \sin(x)e^{-2x}$$

centered around $x = 0$, and use Taylor's Inequality to determine this approximation's maximum error. on $[-1, 1]$.

We begin by determining the first three terms of the TS approximation.

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$\frac{f^{(n)}(0)}{n!}x^n$
0	$\sin(x)e^{-2x}$	0	0
1	$\cos(x)e^{-2x} - 2\sin(x)e^{-2x}$	1	x
2	$-\sin(x)e^{-2x} - 2\cos(x)e^{-2x} - 2\cos(x)e^{-2x} + 4\sin(x)e^{-2x}$ $= -4\cos(x)e^{-2x} + 3\sin(x)e^{-2x}$	-4	$-2x^2$
3	$4\sin(x)e^{-2x} + 8\cos(x)e^{-2x} + 3\cos(x)e^{-2x} - 6\sin(x)e^{-2x}$ $= -2\sin(x)e^{-2x} + 11\cos(x)e^{-2x}$	11	$\frac{11}{6}x^3$

The first three terms are

$$T_3(x) = x - 2x^2 + \frac{11}{6}x^3.$$

(These could also have been found by using the individual functions' Maclaurin Series.)

Taylor's Inequality states that

$$|R_3(x)| \leq \frac{M|x|^4}{4!}, \quad M \geq |f^{(4)}(x)|.$$

$$f^{(4)}(x) = -2\cos(x)e^{-2x} + 4\sin(x)e^{-2x} - 11\sin(x)e^{-2x} - 22\cos(x)e^{-2x} = -24\cos(x)e^{-2x} - 7\sin(x)e^{-2x}$$

$|f^{(4)}(x)| = 24\cos(x)e^{-2x} + 7\sin(x)e^{-2x} \leq 24e^{-2x} + 7e^{-2x} \leq 31e^2$ at $x = -1$, so $M = 31e^2$.

Thus, $|R_3(x)| \leq \frac{31e^2 \cdot (1)^4}{4!} = \boxed{\frac{31}{24}e^2}$ is the maximum error on $[-1, 1]$.

(5) Determine the interval of convergence of the Taylor Series

$$\sum_{n=0}^{\infty}(-1)^n\frac{(x-1)^{4n+1}}{(2n)!}.$$

Then, determine the function modeled by this Taylor Series within the interval of convergence. [Hint: Integrating the series might help!]

We find the interval of convergence through the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(x-1)^{4(n+1)+1}/(2(n+1))!}{(x-1)^{4n+1}/(2n)!}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(x-1)^{4n+5}}{(x-1)^{4n+1}} \cdot \frac{(2n)!}{(2n+2)!}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = (x-1)^4 \cdot \frac{1}{(2n+2)(2n+1)}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = (x-1)^4 \cdot \frac{1}{\infty} = 0,$$

so the interval of convergence is $\boxed{(-\infty, \infty).}$

As the question suggests, we should integrate the series:

$$\int \sum_{n=0}^{\infty}(-1)^n\frac{(x-1)^{4n+1}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{(x-1)^{4n+2}}{(4n+2)(2n!)} = \sum_{n=0}^{\infty}(-1)^n\frac{((x-1)^2)^{2n+1}}{2(2n+1)(2n)!}$$

$$= \frac{1}{2}\sum(-1)^n\frac{((x-1)^2)^{2n+1}}{(2n+1)!},$$

which is very similar to the Taylor Series for $\sin(x)$. In fact, it is the taylor series for $\frac{1}{2}\sin((x-1)^2)$. Thus, because $\int f(x) = \frac{1}{2}\sin((x-1)^2)$, we can see that $\boxed{f(x) = (x-1)\cos((x-1)^2)}$.

(6) (Extra Credit) Prove that if $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ converges then $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges.

Note that $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n} = \sum_{n=1}^{\infty} \sqrt{\frac{a_n}{n^2}}$. We can see that none of our tests work great because a_n is in the sequence, but the information that $\sum_{n=1}^{\infty} a_n$ converges hints at the fact that we should compare $\frac{\sqrt{a_n}}{n}$ to $a_n + f(n)$ somehow, where $\sum_{n=1}^{\infty} f(n)$ converges as well.

We select $f(n) = \frac{1}{n^2}$. Notice that

$$\frac{\sqrt{a_n}}{n} \overset{?}{\leq} a_n + \frac{1}{n^2}$$

$$\sqrt{\left(\frac{a_n}{n^2}\right)} \overset{?}{\leq} a_n + \frac{1}{n^2}$$

$$\frac{a_n}{n^2} \overset{?}{\leq} a_n^2 + \frac{2a_n}{n^2} + \frac{1}{n^4}$$

$$0 \overset{?}{\leq} a_n^2 + \frac{a_n}{n^2} + \frac{1}{n^4}.$$

All of the terms on the right are positive because $a_n \geq 0$, so the inequality must hold for all n. Thus

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n} \leq \sum_{n=1}^{\infty} \left(a_n + \frac{1}{n^2}\right)$$

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n} \leq \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} \frac{1}{n^2}$$

and both series on the right converge, so by the Comparison Test, the series on the left must converge as well.