CLASSIFYING VECTOR FIELDS

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1. Question

Say we are given a vector field $\mathbf{F} = \langle P, Q \rangle$. What is the best way to determine whether or not it is conservative?

2. Answer

Almost always, the best way to do this would be to use the following theorem:

Theorem 2.1. *Given a vector field* $\mathbf{F} = \langle P, Q \rangle$*, if*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},$$

then $\mathbf{F}$ *is conservative.*

The reason we love this theorem is that it's super simple to use (just differentiate twice), so we use it basically whenever we can to prove that fields are conservative. Conservative vector fields have the property that all line integrals on the field are path-independent, but there's no easy way to prove a statement like that.

Try not to get too caught up in the terminology of things like simple curves and open regions; while they're definitely necessary mathematically to make the definition more specific, they're not all too directly relevant to our course material and won't help much with this.

There are only two reasons I can think of to not use this method:

(1) **We're almost certain that the field is conservative, and need to integrate something over it.** In this case, it might be faster to just go ahead and look for the potential function f for this vector field. After all, the existence of this potential function would prove that the field is conservative, and we'd use f to compute the line integral anyway, so it'd save us a step if we just began by finding f. To find f, simply find the antiderivatives of P and Q with respect to their term on the gradient.

(2) **We're not given the equation, only the vector field drawn on a graph.** Now, this is the case where we'll have to get more intuitive with our approach. We'll rely in this case on the fact that in any conservative vector field, any line integral that travels in a closed loop in the vector field will evaluate to exactly 0. I'll begin by stating this formally, and then talk about how to informally think about it (or at least how I think about it).

The property that we want to use is that for any closed simple (read: not self-intersecting) loop C within the vector field, if $\mathbf{F}$ is conservative then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

I believe Professor Hutchings proves this formally in one of his lectures.

Anyway, the good thing about potential functions and conservative fields is that they actually have super intuitive ties to the real world! After all, gravity is a conservative vector field, which we should have a great intuition with. For this reason, when I think about these problems I like to think about it in terms of gravity. In particular, I like to think of the field as being on a hill, where the arrows always point in the direction where gravity goes, which is directly downhill. If I were to start at some point on the hill, walk in a circle, and arrive back at the same point, I should have been travelling downhill around half of the time and uphill the other half, right? At least, I certainly shouldn't be able to take a circular path where I'm always going downhill and end up in the same place.

That's the test I use for problems like those. I'd literally just draw a couple of circles on the graph, and for each one I'd check that my circle is sometimes going downhill (kinda in the direction of the arrows) and is sometimes going uphill (kinda against the direction of the arrows). If I can ever find a path that is always downhill or always uphill, then I know that my hill is not on a conservative field.

Unfortunately, using this type of approach will definitely take some intuition and some ability to visualize things that aren't necessarily the original intent of the graph, so take this with a grain of salt. That being said, as far as I can tell that's the best way to determine using only a graph whether a vector field is conservative. I would suspect that any question they give you like that should make it fairly easy to tell using a method like this.