FTLI APPLICATION

JONATHAN LIU

1. Question

Let f be a differentiable function on $\mathbb{R}^2$ such that

$$\frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial y}(x,y)$$

for all x, y. Suppose that $f(2,3) = 6$. Compute $f(4,1)$. As usual, justify your answer.

2. Answer

We want to know $f(4,1)$, and we know $f(2,3)$. Now, we don't have many tools to relate two different values of a function, so our options for how to proceed here are pretty limited. In this case, we can realize that the Fundamental Theorem of Line Integrals is actually helpful because it tells us the difference between f at two different points! In particular, remember that it tells us for any conservative field $\mathbf{F} = \nabla f$ and curve C from a to b,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(b) - f(a).$$

Using this, we'll select some curve C' that has endpoints at $(2,3)$ and $(4,1)$ and any parametrization $\mathbf{r}$ that we want. In this case, we have that

$$\int_{C'} (\nabla f) \cdot d\mathbf{r} = f(4,1) - f(2,3).$$

Now, expanding and rewriting this integral gives us

$$\begin{aligned}\int_{C'} (\nabla f) \cdot d\mathbf{r} &= \int_{C'} \left\langle \frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right\rangle \cdot (x'(t), y'(t))\, dt \\ &= \int_{C'} \frac{\partial f}{\partial x}(x,y)\, dx + \int_{C'} \frac{\partial f}{\partial y}(x,y)\, dy\end{aligned}$$

However, note that $\frac{\partial f}{\partial x}(x,y) = \frac{\partial f}{\partial y}(x,y)$ and the total change in x is -2 while the total change in y is 2, so the two integrals have the same integrand and directly opposite bounds, so their values are opposite. Their sum is therefore zero, so the integral evaluates to zero. This means that

$$f(4,1) - f(2,3) = 0,$$

so $f(4,1) = f(2,3)$.

2.1. Alternate Solution.

The second solution basically uses the parameterization $x = t$, $y = 5 - t$ and attempts to integrate on that line from one point to the other. When it does so, it has to take

$$\frac{df}{dt} = \left\langle \frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle = \left\langle \frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial x}(x,y) \right\rangle \cdot \langle 1, -1 \rangle = 0,$$

so the line integral evaluates to zero. From there, we use the same FTLI logic.