PROVING THE EXISTENCE OF A LIMIT

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1. Question

Find and prove the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}.$$

2. Answer

The good news is that there are many options to prove this limit. The bad news is that none of them are particularly easy to see/do, but I've shown the three main methods here.

2.1. **Proof 1: Conversion to Polar.** The term $\sqrt{x^2+y^2}$ in the denominator might be a hint that polar coordinates would be helpful, so let's try converting this limit to polar coordinates. Using $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} &= \lim_{(r,\theta)\to(0,0)} \frac{(r\cos\theta)(r\sin\theta)}{\sqrt{r^2}} \\
&= \lim_{(r,\theta)\to(0,0)} r\sin\theta\cos\theta \\
&= \left(\lim_{r\to 0} r\right)\left(\lim_{\theta\to 0} \frac{1}{2}\sin(2\theta)\right) \\
&= 0.
\end{aligned}$$

2.2. **Proof 2: Squeeze Theorem.** Squeeze Theorem certainly works as a viable option, but you'll have to be more clever about the function you use to squeeze. In particular, we want to find a relationship between the numerator and denominator of the fraction that allows us to compare them and get rid of the denominator. After all, once we stop dividing by 0, it'll be much easier to evaluate the limit. The fact that there's an x^2+y^2 term as well as an xy term might hint to something like $x^2 - 2xy + y^2 = (x-y)^2$ (it's definitely not easy to see, but it's there). In particular, note that because any squared number is non-negative, we have

$$\begin{aligned}
(x-y)^2 &\geq 0 \\
(x+y)^2 &\geq 0 \\
x^2 - 2xy + y^2 &\geq 0 \\
x^2 + 2xy + y^2 &\geq 0 \\
x^2 + y^2 &\geq |2xy| \\
\sqrt{x^2+y^2} &\geq \sqrt{|2xy|},
\end{aligned}$$

so we have

$$\left|\frac{xy}{\sqrt{x^2+y^2}}\right| \leq \frac{|xy|}{\sqrt{2|xy|}},$$

and we can squeeze the absolute value of a function with 0 to get

$$\begin{aligned}
0 \leq \left|\frac{xy}{\sqrt{x^2+y^2}}\right| &\leq \frac{|xy|}{\sqrt{2|xy|}} \\
\lim_{(x,y)\to(0,0)} 0 \leq \lim_{(x,y)\to(0,0)} \left|\frac{xy}{\sqrt{x^2+y^2}}\right| &\leq \lim_{(x,y)\to(0,0)} \frac{|xy|}{\sqrt{2|xy|}} \\
0 \leq \lim_{(x,y)\to(0,0)} \left|\frac{xy}{\sqrt{x^2+y^2}}\right| &\leq \lim_{(x,y)\to(0,0)} \frac{\sqrt{|xy|}}{\sqrt{2}} \\
0 \leq \lim_{(x,y)\to(0,0)} \left|\frac{xy}{\sqrt{x^2+y^2}}\right| &\leq 0,
\end{aligned}$$

which completes our squeeze. You'll note that we purposefully squeezed with the absolute value rather than with the original function, which works great for functions whose limit is 0 because if a function's absolute value approaches 0 then the function itself must always approach 0 as well.

2.3. **Proof 3: Definition of a Limit.** Remember the formal definition of a limit in two variables: a function f has limit L at (a,b) if for every number $\epsilon > 0$ there is a corresponding $\delta > 0$ such that for all (x,y) if $\sqrt{(x-a)^2+(y-b)^2} \leq \delta$ then $|f(x,y)-L| \leq \epsilon$. Thus, we just need to be able to find a δ for every ϵ such that $|f(x,y)| < \epsilon$. To do so, we'll try to find a way to compare $|f(x,y)|$ to $\delta = \sqrt{x^2+y^2}$.

$$\begin{aligned}
|f(x,y)-0| &= \frac{|xy|}{\sqrt{x^2+y^2}} \\
&= \frac{\left(\sqrt{x^2}\right)\left(\sqrt{y^2}\right)}{\sqrt{x^2+y^2}} \\
&\leq \frac{\left(\sqrt{x^2+y^2}\right)\left(\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}} \\
&\leq \sqrt{x^2+y^2} \\
&= \delta,
\end{aligned}$$

so we know that $|f(x,y)-0| \leq \delta$. Thus, for every $\epsilon > 0$, we can set $\delta = \epsilon$ which ensures that $|f(x,y)| \leq \epsilon$. Because this is possible, the limit of f at $(0,0)$ is 0.